



GURU NANAK ACADEMY SAFIDON (9729630333)

gurunanakmaths@gmail.com

QUESTIONS FOR PRACTICE(CLASS - XII)

INVERSE TRIGNOMETRY

1. Prove the following: $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$
2. Prove that: $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$.
3. Prove that $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$
4. Prove the following: $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right)$
5. Prove that: $\tan^{-1}\left(\frac{62}{16}\right) = \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5}$
6. $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$
7. Prove the following: $\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$
8. Prove the following: $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{8}{19}\right) = \frac{\pi}{4}$.
9. Solve for x : $2\tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$
10. Solve for x: $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}, x > 0$.

SOLUTION

INVERSE TRIGNOMETRY

$$1. \tan^{-1} \frac{2/3}{1-1/3^2} + \tan^{-1} \frac{1}{7}$$

$$\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{7} \rightarrow \tan^{-1} 1 = \frac{\pi}{4}$$

$$2. \text{LHS} = \tan^{-1}\left(\frac{3}{-1}\right) + \tan^{-1} 3$$

$$= \tan^{-1}\left(\frac{-3+3}{1+9}\right) = \tan^{-1}(0) = \pi$$

$$3. \tan^{-1}\frac{7}{9} + \tan^{-1}\frac{1}{8} = \tan^{-1}\frac{56+9}{72-7} = \tan^{-1} 1 = \frac{\pi}{4}$$

$$4. 2 \tan^{-1}\frac{\frac{1}{4} + \frac{2}{9}}{1 - \frac{1 \cdot 2}{4 \cdot 9}} = 2 \tan^{-1}\frac{17}{34} = 2 \tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{1 - \frac{1}{4}} = \sin^{-1}\frac{4}{5}$$

$$5. \text{RHS} = \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{4}{3} = \tan^{-1}\frac{15+48}{36-20} = \text{LHS}$$

$$6. 1 + \sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} = \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2$$

$$1 - \sin x = \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2$$

$$\text{LHS} = \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right) = \cot^{-1}\left(\cot\frac{x}{2}\right) = x/2$$

$$7. \text{LHS} = \tan^{-1}\left(\frac{15+12}{20-9}\right) - \tan^{-1}\left(\frac{8}{19}\right)$$

$$= \tan^{-1}\left(\frac{27}{11}\right) - \tan^{-1}\left(\frac{8}{19}\right) = \tan^{-1}\left(\frac{513-88}{209+216}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$8. \tan^{-1}\left(\frac{2\cos x}{1 - \cos^2 x}\right) = \tan^{-1}(\csc x)$$

$$\left(\frac{2\cos x}{\sin^2 x}\right) = \frac{2}{\sin^2 x} \rightarrow \sin x(\cos x - \sin x) = 0$$

$$x = 0, x = \frac{\pi}{4}, x = 0 \text{ satisfies the given equation}$$

$$9. \tan^{-1}\frac{2x+3x}{1-2x \cdot 3x} = \frac{\pi}{4} \rightarrow \tan^{-1}\frac{5x}{1-6x^2} = \frac{\pi}{4} \rightarrow \frac{5x}{1-6x^2} = 1 \rightarrow (6x-1)(x+1) = 0$$

$$\rightarrow x = \frac{1}{6}, -1 \rightarrow x = \frac{1}{6} \text{ as } x > 0$$

$$10. \tan^{-1} \frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)} = \frac{\pi}{4} \rightarrow \frac{2x^2-4}{-3} = 1 \rightarrow x = \pm \frac{1}{\sqrt{2}}$$

Find the values/ relation of a, b, c and k wherever required

$$11. f(x) = \begin{cases} ax+1, & \text{if } x \leq 3 \\ bx+3, & \text{if } x > 3 \end{cases} \text{ is continuous at } x=3$$

$$12. f(x) = \begin{cases} k(x^2-2x), & x \leq 0 \\ 4x+1, & x > 0 \end{cases} \text{ is continuous at } x=0 \text{ also discuss about continuity at } x=1.$$

$$13. f(x) = \begin{cases} k \cos x, & \text{if } x \neq \frac{\pi}{2} \\ \pi - 2x, & \text{if } x = \frac{\pi}{2} \end{cases} \text{ is continuous at } x = \frac{\pi}{2}$$

$$1. f(x) = \begin{cases} kx+1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases} \text{ is continuous at } x=\pi$$

$$14. f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax+b, & 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases} \text{ is continuous function.}$$

$$15. \text{ Show that } f(x) = |x-3| \text{ is continuous but not differentiable at } x=3$$

$$16. f(x) = \begin{cases} \frac{1-\cos x}{kx^2}, & \text{if } x \neq 0 \\ 3, & \text{if } x = 0 \end{cases} \text{ is continuous at } x=0$$

$$17. f(x) = \begin{cases} \frac{x^5 - 2^5}{x - 2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases} \text{ is continuous at } x=2$$

$$18. f(x) = \begin{cases} \frac{\sin(x-2)}{x-2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases} \text{ is continuous at } x=2$$

Find, $\frac{d^2y}{dx^2}$ when a function is given by

$$19. x = \sin t, y = \cos 2t$$

$$20. x = \cos t, y = (1 + \cos t)$$

$$21. x = a(\cos t + \log(\tan \frac{t}{2})), y = a \sin t$$

$$22. x = 2at^2, y = at^4$$

$$23. x = 4t, y = \frac{4}{t} \text{ find, } \frac{dy}{dx} \text{ when a function is given by}$$

$$24. x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

$$25. \text{ if } x\sqrt{1+y} + y\sqrt{1+x} = 0 \text{ then prove that } \frac{dy}{dx} = \frac{-1}{(1+x)^2}$$

$$26. \text{ if } (x-a)^2 + (y-b)^2 = c^2 \text{ then prove that } \frac{[1 + (\frac{dy}{dx})^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \text{ is independent of } a \text{ and } b$$

$$27. \text{ if } \cos y = x \cos(a+y) \text{ then prove that } \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

$$28. \text{ if } y = e^{a \cos^{-1} x} \text{ then show that } (1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$$

Solutions: continuity and differentiability

11. As $f(x) = \begin{cases} ax+1, & \text{if } x \leq 3 \\ bx+3, & \text{if } x > 3 \end{cases}$ is continuous at $x=3$

$$\therefore \lim_{x \rightarrow 3^+} f(x) = f(3)$$

$$3b+3 = 3a + 1$$

$$a-b = 2/3$$

12. as $f(x) = \begin{cases} k(x^2-2x), & x \leq 0 \\ 4x+1, & x > 0 \end{cases}$ is continuous at $x=0$

$$\therefore \lim_{x \rightarrow 0^+} f(x) = f(0)$$

$$1 = k(0)$$

$1=0$ which is not possible for any value of k

$$\lim_{x \rightarrow 1} f(x) = f(1)$$

$$5=5 \quad \therefore \text{for all values of } k \text{ } f(x) \text{ is continuous at } x = 1$$

13. $f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{k \sin(\frac{\pi}{2} - x)}{2(\frac{\pi}{2} - x)} = 3$$

$$k/2 = 3 \quad k = 6$$

14. as $f(x) = \begin{cases} kx+1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$ is continuous at $x = \pi$

$$\therefore \lim_{x \rightarrow \pi^+} f(x) = f(\pi)$$

$$\lim_{x \rightarrow \pi^+} \cos x = k\pi + 1$$

$$-1 = k\pi + 1 \quad K = -2/\pi$$

15. as $f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax+b, & 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases}$ is continuous function.

$$\therefore \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$\therefore \lim_{x \rightarrow 2^+} ax+b = 5$$

$$2a+b = 5 \dots (1)$$

$$\therefore \lim_{x \rightarrow 10^-} f(x) = f(10)$$

$$\therefore \lim_{x \rightarrow 10^-} ax+b = 21$$

$$10a+b = 21 \dots (2)$$

$$a=2, b=1$$

$$16. f(x) = \begin{cases} -x+3 & \text{if } x < 3 \\ x-3, & \text{if } x \geq 3 \end{cases} \quad \text{at } x=3$$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3)$$

$$\lim_{h \rightarrow 0} -(3-h)+3 = \lim_{h \rightarrow 0} (3+h)-3 = 0$$

$$0=0=0$$

\therefore continuous at $x=3$

$$\therefore \text{L.H.D} = \lim_{h \rightarrow 0} \frac{f(3-h)-f(3)}{-h} \text{H.D} = \lim_{h \rightarrow 0} \frac{f(3-h)-f(3)}{-h} = \lim_{h \rightarrow 0} \frac{-3+h+3}{-h} = -1$$

$$\text{R.H.D} = \lim_{h \rightarrow 0} \frac{f(3+h)-f(3)}{h} = \lim_{h \rightarrow 0} \frac{3+h-3}{h} = 1$$

As $\text{L.H.D} \neq \text{R.H.D}$

$\therefore f(x)$ is not differentiable at $x=3$.

$$17. \text{As } f(x) = \begin{cases} \frac{1-\cos x}{kx^2}, & \text{if } x \neq 0 \\ 3, & \text{if } x = 0 \end{cases} \text{ is continuous at } x=0$$

$$\lim_{x \rightarrow 0} f(x) = f(0)$$

$$\lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{4k \frac{x}{4}} = 3$$

$$\frac{1}{2k} = 3$$

$$K = 1/6$$

$$18. \text{ As } f(x) = \begin{cases} \frac{x^5 - 2^5}{x - 2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases} \text{ is continuous at } x = 2$$

$$\therefore \lim_{x \rightarrow 2} f(x) = f(2)$$

Question based on rate of change.

29. The side of a square is changing at the rate of 5cm/sec. How fast is its area increasing when the Length of its side is 8cm.
30. The rate of change of the side of a cube 6 cm /sec .How fast is its surface area changing when the length of its edge is 5 cm?
31. The volume of a cube is increasing at the rate $10 \frac{\text{cm}^3}{\text{sec}}$. How fast is the surface area increasing when the length of an edge is 15 cm?
32. Find the point on the curve $y^2 = 3x$ for which the abscissa and ordinate change at the same rate.
33. An edge of a variable cube is increasing at the rate of 3cm/s. How fast is the volume of The cube increasing when the edge is 10 cm long.
34. A particle move along the curve $6y = x^3 + 2$. Find the points on the curve at which the

Y coordinate is changing 8 times as fast as the x –coordinate.

35. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?
36. Aman 2m high, walks at a uniform speed of 6m/min. away from a lamp post 5m high. Find the rate at which the length of his shadow increases.
37. The radius of a circle is increasing uniformly at the rate of 3cm/s. find the rate at which the area of the circle is increasing when the radius is 10cm.
38. Sand is pouring from a pipe at the rate of $12 \frac{\text{cm}^3}{\text{sec}}$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one –sixth of the Radius of the base. How fast is the height of the sand cone increasing when the Height is 4cm?

SOLUTIONS

Q.29 Let side of square at any time be x cm.

$$\text{The } \frac{dx}{dt} = 5 \text{ cm/sec}$$

$$\text{Now, area of square, } A = x^2$$

$$\therefore \text{The } \frac{dA}{dt} = 80 \text{ cm}^2/\text{s} \text{ Ans.}$$

Q. 30 Let x be the length of the cube and s be the surface area at any time t

$$\frac{dx}{dt} = 6 \text{ cm/sec}$$

$$s = 6x^2$$

$$\text{The } \frac{ds}{dt} = 360 \text{ cm}^2/\text{s}$$

Q. 31 Here $6y = x^3 + 2$ and $\frac{dy}{dt} = 8\frac{dx}{dt}$

Solving them we get $x = \pm 4$

At $x = 4$ $y = 11$

At $x = -4$, $y = -\frac{32}{3}$

Thus required points are $(4, 11)$ and $(-4, -\frac{32}{3})$

Q. 32 Let r be the radius of bubble and v be the volume of bubble at any time t .

Then $\frac{dr}{dt} = \frac{1}{2} \frac{\text{cm}}{\text{s}}$ and $r = 1 \text{ cm}$

By solving we get $\frac{dv}{dt} = 2\pi \text{ cm}^3/\text{s}$

Q. 33 Let AB is the Lamp post.

At any time t , let the man be at a distance of x m away from the lamp post and

Y m be length of his shadow

Then $AB = 5 \text{ m}$ and $CD = 2 \text{ m}$

$\frac{dx}{dt} = 6 \text{ m/min}$

By solving we get $\frac{dy}{dt} = 4 \text{ m/min}$

Q. 34 Let r be the radius and A be the area of the circle at any time t .

Now $\frac{dr}{dt} = 3 \text{ cm/s}$ and $r = 10 \text{ cm}$

Now $A = \pi r^2$

By solving it we get $\frac{dA}{dt} = 60\pi \text{ cm}^2/\text{s}$

Q. 35 Let r be the radius h be the height and v be the volume of sand

Cone at any time t .

Then $\frac{dv}{dt} = 12 \text{ cm}^3/\text{s}$ and $h = \frac{1}{6} r$

By solving we get $\frac{dh}{dt} = \frac{1}{48\pi}$ cm/sec

$$\lim_{x \rightarrow 2} \frac{x^5 - 2^5}{x - 2} = k$$

$$5(2^4) = k$$

$$K = 80$$

$$36. \text{ As } f(x) = \begin{cases} \frac{\sin(x-2)}{x-2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases} \text{ is continuous at } x=2$$

$$\therefore \lim_{x \rightarrow 2} f(x) = f(2)$$

$$\lim_{x \rightarrow 2} \frac{\sin(x-2)}{x-2} = k$$

$$K = 1$$

$$37. \text{ As } f(x) = \begin{cases} \frac{e^x - 1}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases} \text{ is continuous at } x=0$$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = k$$

$$1 = k$$

Differentiation:

$$19, \quad x = \sin t, \quad y = \cos 2t$$

$$\frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = -2\sin 2t$$

$$\therefore \frac{dy}{dx} = -2\sin 2t / \cos t$$

$$\therefore \frac{dy}{dx} = -4 \sin t$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{4\sin t}{\cos t} = -4 \tan t$$

20. $x = \cos t - \cos 2t, y = (1 + \cos t)$

$$\therefore \frac{dx}{dt} = 2\sin 2t - \sin t, \quad \frac{dy}{dt} = -\sin t$$

$$\therefore \frac{dy}{dx} = \frac{1}{1 - 4 \cos t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-4\sin t}{(1 - 4\cos t)^2} * \frac{1}{\sin t(4\cos t - 1)} = \frac{1}{(1 - 4\cos t)^3}$$

21 $x = a(\cos t + \log(\tan \frac{t}{2})), y = a \sin t$

$$\therefore \frac{dx}{dt} = a(-\sin t + \frac{1}{\sin t}) = a \frac{\cos^2 t}{\sin t}$$

$$\therefore \frac{dy}{dt} = a \cos t$$

$$\therefore \frac{dy}{dx} = \tan t$$

$$\therefore \frac{d^2y}{dx^2} = \sec^2 t * \frac{\sin t}{a \cos^2 t}$$

22 $x = 2at^2, y = at^4$

$$\frac{dx}{dt} = 4at, \frac{dy}{dt} = 4at^3$$

$$\frac{dy}{dx} = t^2$$

$$\frac{d^2y}{dx^2} = 2t \cdot \frac{1}{4at}$$

$$23 \quad x = 4t, y = \frac{4}{t}$$

$$\frac{dx}{dt} = 4, \frac{dy}{dt} = -\frac{4}{t^2}$$

$$\frac{dy}{dx} = -\frac{1}{t^2}$$

$$\frac{d^2y}{dx^2} = \frac{2}{t^3} \cdot \frac{1}{4}$$

$$24 \quad x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

$$\frac{dx}{dt} = \frac{\sin^2 t \cos t (2 \cos 2t + 1)}{(\cos 2t)^{3/2}}, \frac{dy}{dt} = \frac{\cos^2 t \sin t (1 - 2 \cos 2t)}{(\cos 2t)^{3/2}}$$

$$\frac{dy}{dx} = \cot t$$

$$25 \quad x\sqrt{1+y} + y\sqrt{1+x} = 0$$

$$x\sqrt{1+y} = -y\sqrt{1+x}$$

$$y = \frac{-x}{1+x}$$

$$\frac{dy}{dx} = \frac{-1}{(1+x)^2}$$

$$(x-a) + (y-b) \frac{dy}{dx} = 0$$

$$26 \quad (x-a)^2 + (y-b)^2 = c^2$$

$$1 + (y-b) \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\frac{d^2y}{dx^2} = -(1+y_1^2)$$

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = - \frac{(1+y_1^2)^{3/2}}{(1+y_1^2)} = -(1+y_1^2)^{1/2} = -(c^2)^{1/2} = -c$$

$$27. \text{ if } \cos y = x \cos(a+y) \text{ then prove that } \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

$$X = \frac{\cos y}{\cos(a+y)}$$

$$\frac{dx}{dy} = \frac{-\sin y \cos(a+y) + \sin(a+y) \cos y}{\sin^2(a+y)} \cdot \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

$$28. \text{ if } y = e^{\cos^{-1}x} \text{ then show that } (1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$$

$$\frac{dy}{dx} = e^{\cos^{-1}x} \cdot \frac{-a}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} \frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}} \frac{dy}{dx} = e^{\cos^{-1}x} \cdot \frac{-a^2}{\sqrt{1-x^2}}$$

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$$

INCREASING AND DECREASING FUNCTIONS

39. Find the intervals in which the function f given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is strictly increasing .

40. Find the intervals in which the function f given by $f(x) = -2x^3 - 9x^2 - 12x + 1$ is strictly decreasing .
41. Find the intervals in which the function f given by $f(x) = \sin x + \cos x$, $0 \leq x \leq 2\pi$ is strictly increasing .
42. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is strictly increasing and decreasing.
43. Show that the function $f(x) = e^{2x}$ is strictly increasing on \mathbb{R} .
44. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x + 30$ is strictly increasing.
45. Prove that the logarithmic function is strictly increasing on $(0, \infty)$.
46. Find the value of x for which $y = [x(x-2)]^2$ is an increasing function.
47. Prove that the function given by $f(x) = \cos x$ is (a) strictly decreasing in $(0, \pi)$.
(b) strictly increasing in $(\pi, 2\pi)$. (c) neither increasing nor decreasing in $(0, 2\pi)$.

TANGENTS AND NORMALS

48. Find the equations of the tangent to the curve $y = x^2 + 4x + 1$ at the point whose x -coordinate is 3.
49. Find the equations of the normal to the curve $y = x^2 + 4x + 1$ at the point whose x -coordinate is 3.
50. Find the slope of the normal to the curve $y = x^2 - 3x + 2$ at the point whose x -coordinate is 3.
51. Find the equations of the tangent to the curve $x = \cos t$, $y = \sin t$ at $t = \frac{\pi}{4}$.
52. Find the equation of the normal at the point $(1,1)$ on the curve $2y + x^2 = 3$.
53. Find the equation of tangent to the curve $y = \sqrt{3x-2}$, which is parallel to the line $4x - 2y + 5 = 0$.
54. Find the equations of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.
55. Find the equations of the tangent to the parabola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (h,k) .
56. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

57. On which point the line $y = x + 1$ is a tangent to the curve $y^2 = 4x$

SECOND ORDER DERIVATIVE

58. If $y = a \sin x + b \cos x$, then prove that $\frac{d^2y}{dx^2} + y = 0$.

59. If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0$.

60. If $y = 500e^{7x} + 600e^{-7x}$, prove that $\frac{d^2y}{dx^2} - 49y = 0$.

61. Find $\frac{d^2y}{dx^2}$, if $y = x^3 + \tan x$.

62. If $y = 3 \cos (\log x) + 4 \sin (\log x)$, show that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$.

63. Find $\frac{d^2y}{dx^2}$, if $x = a \cos t$, $y = b \sin t$.

64. If $f(x) = x^3 - 3x^2 + 3x + 30$, then find $f''(-3)$.

65. Find the second order derivative of $x^3 + \log x$.

66. If $y = \tan^{-1} x$, prove that $(1 + x^2) y_2 + 2x y_1 = 0$.

67. If $y = \sin^{-1} x$, prove that $(1 - x^2) y_2 - x y_1 = 0$.

Solutions

INCREASING AND DECREASING FUNCTIONS

39. by $f(x) = 4x^3 - 6x^2 - 72x + 30$

$$f'(x) = 12x^2 - 12x - 72 = 12(x^2 - x - 6) = 12(x-3)(x+2)$$

Critical points are -2,3.

Interval	$x-3$	$x+2$	$12(x-3)(x+2)$	Nature
$x < -2$	-	-	+	increasing
$-2 < x < 3$	-	+	-	Decreasing

$x > 3$	+	+	+	increasing
---------	---	---	---	------------

40. $f(x) = -2x^3 - 9x^2 - 12x + 1$

$$f'(x) = -6x^2 - 18x - 12 = -6(x^2 + 3x + 2) = -6(x+1)(x+2)$$

Critical points are -1, -2.

Interval	$x+1$	$x+2$	$-6(x+1)(x+2)$	Nature
$x < -2$	-	-	-	Decreasing
$-2 < x < -1$	-	+	+	increasing
$x > -1$	+	+	-	Decreasing

41. $f(x) = \sin x + \cos x, 0 \leq x \leq 2\pi$

$$f'(x) = \cos x - \sin x,$$

$$f'(x) = 0, \text{ gives } \cos x = \sin x,$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

Interval	Sign of $f'(x)$	Nature
$[0, \frac{\pi}{4})$	+	increasing
$(\frac{\pi}{4}, \frac{5\pi}{4})$	-	Decreasing
$(\frac{5\pi}{4}, 2\pi)$	+	increasing

42. $f(x) = 2x^3 - 3x^2 - 36x + 7$

$$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x-3)(x+2)$$

Critical points are -2, 3.

Interval	$x-3$	$x+2$	$6(x-3)(x+2)$	Nature
----------	-------	-------	---------------	--------

			(x+2)	
$x < -2$	-	-	+	increasing
$-2 < x < 3$	-	+	-	Decreasing
$x > 3$	+	+	+	increasing

43. $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

Since e^{2x} is always positive . Therefore $f'(x) > 0$

44. $f(x) = x^3 - 3x^2 + 3x + 30$.

$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2$

Since $(x - 1)^2$ is always positive . Therefore $f'(x) > 0$

45. $f(x) = \log x, x > 0$

$f'(x) = \frac{1}{x},$

$f'(x) > 0$, function is strictly increasing on $(0, \infty)$.

46. $y = [x(x-2)]^2 = x^2(x-2)^2$

$\frac{dy}{dx} = 2x(x-2)^2 + 2x^2(x-2) = 2x(x-2)(x-2+x) = 4x(x-2)(x-1)$

Critical points are 0, 1, 2 .

interval	x	x-1	x-2	$4x(x-2)(x-1)$	Nature
$x < 0$	-	-	-	-	Decreasing
$0 < x < 1$	+	-	-	+	increasing
$1 < x < 2$	+	+	-	-	Decreasing
$x > 2$	+	+	+	+	increasing

47. $f(x) = \cos x$

$f'(x) = -\sin x$

(a) In $(0, \pi)$, $\sin x > 0$, $-\sin x < 0 \Rightarrow f'(x) < 0$.

(b) In $(\pi, 2\pi)$, $\sin x < 0$, $-\sin x > 0 \Rightarrow f'(x) > 0$.

(c) Clearly from (a) and (b), f is neither increasing nor decreasing in $(0, 2\pi)$.

TANGENTS AND NORMALS

48. curve $y = x^2 + 4x + 1$

$\frac{dy}{dx} = 2x + 4$

$\frac{dy}{dx}$ at $x = 3$ is 10. So $m = 10$.

$x = 3, y = 22$

equation of the tangent is

$y - 22 = 10(x - 3)$ i. e. $10x - y = 8$.

49. curve $y = x^2 + 4x + 1$

$\frac{dy}{dx} = 2x + 4$

$\frac{dy}{dx}$ at $x = 3$ is 10.

So slope of normal is $-\frac{1}{10}$.

equation of the normal is

$\frac{1}{10}(x - 3)$ i. e. $x + 10y = 223$

50. $y = x^2 - 3x + 2$

$$\frac{dy}{dx} = 2x - 3.$$

$$\frac{dy}{dx} \text{ at } x = 3 \text{ is } 3.$$

$$\text{So slope of normal is } -\frac{1}{3}.$$

$$51. x = \cos t, y = \sin t$$

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t \quad \Rightarrow \frac{dy}{dx} = -\cot t$$

$$\frac{dy}{dx} \text{ at } t = \frac{\pi}{4} = -1$$

$$x = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad y = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Hence, equation of the tangent is

$$\frac{1}{\sqrt{2}} = -1 \left(x - \frac{1}{\sqrt{2}} \right) \quad \Rightarrow \quad x + y = \sqrt{2}.$$

$$52. \text{ Curve } 2y + x^2 = 3$$

$$2 \frac{dy}{dx} + 2x = 0 \quad \Rightarrow \frac{dy}{dx} = -x \quad \Rightarrow \frac{dy}{dx} \text{ at } (1,1) = -1$$

So slope of normal is 1.

equation of the normal is

$$y - 1 = 1(x - 1) \quad \Rightarrow \quad x - y = 0$$

$$53. \text{ curve } y = \sqrt{3x-2},$$

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}}$$

Slope of the line $4x - 2y + 5 = 0$ is 2.

$$\text{Given } \frac{3}{2\sqrt{3x-2}} = 2 \Rightarrow x = \frac{41}{48} \text{ and } y = \frac{3}{4}$$

Hence, equation of the tangent is the tangent is

$$\frac{3}{4} = 2 \left(x - \frac{41}{48} \right) \Rightarrow 48x - 24y - 23 = 0$$

54. parabola $y^2 = 4ax$

$$2y \frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y} \Rightarrow \frac{dy}{dx} \text{ at the point } (at^2, 2at) = \frac{1}{t}$$

Hence, equation of the tangent is tangent is

$$2at = \frac{1}{t} (x - at^2) \Rightarrow x - ty + at^2 = 0$$

$$55. \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

$$\frac{dy}{dx} \text{ at the point } (h, k) = \frac{b^2h}{a^2k}$$

Hence, equation of the tangent is

$$y - k = \frac{b^2h}{a^2k} (x - h) \Rightarrow \frac{hx}{a^2} - \frac{ky}{b^2} = 1$$

$$56. x = y^2 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

$$\text{and } xy = k \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

$$\text{cut at right angles if } \frac{1}{2y} x \frac{-y}{x} = -1 \text{ so } x = \frac{1}{2}, y^2 = \frac{1}{2}$$

$$\text{Since } xy = k, \Rightarrow x^2 y^2 = k^2 \Rightarrow \left(\frac{1}{2}\right)^2 \frac{1}{2} = k^2$$

$$\Rightarrow 8k^2 = 1$$

$$57. y^2 = 4x \Rightarrow 2y \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

Slope of the line $y = x + 1$ is 1.

$$\Rightarrow \frac{2}{y} = 1 \Rightarrow y = 2. \text{ Then } x = 1.$$

\Rightarrow So required point is $(1, 2)$.

SECOND ORDER DERIVATIVE

$$58. y = a \sin x + b \cos x,$$

$$\frac{dy}{dx} = a \cos x - b \sin x$$

$$\frac{d^2y}{dx^2} = -a \sin x - b \cos x = -y$$

$$\frac{d^2y}{dx^2} + y = 0.$$

$$59. y = 3e^{2x} + 2e^{3x},$$

$$\frac{dy}{dx} = 6e^{2x} + 6e^{3x}$$

$$\frac{d^2y}{dx^2} = 12e^{2x} + 18e^{3x},$$

On substituting we get, $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$

$$60. y = 500e^{7x} + 600e^{-7x},$$

$$\frac{dy}{dx} = 7(500e^{7x} - 600e^{-7x})$$

$$\frac{d^2y}{dx^2} = 49(500e^{7x} + 600e^{-7x}) = 49y$$

61. $y = x^3 + \tan x$.

$$\frac{dy}{dx} = 3x^2 + \sec^2 x$$

$$\frac{d^2y}{dx^2} = 6x + 2 \sec^2 x \cdot \tan x$$

62. $y = 3 \cos (\log x) + 4 \sin (\log x)$,

$$\frac{dy}{dx} = -3 \sin \log x \cdot \frac{1}{x} + 4 \cos \log x \cdot \frac{1}{x}$$

$$x \frac{dy}{dx} = -3 \sin \log x + 4 \cos \log x$$

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} = -(3 \cos (\log x) + 4 \sin (\log x)) = -y$$

$$\text{Hence, } x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0.$$

63. $x = a \cos t$, $y = b \sin t$.

$$\frac{dx}{dt} = -a \sin t, \quad \frac{dy}{dt} = b \cos t$$

$$\frac{dy}{dx} = -\frac{b}{a} \cot t$$

$$\frac{d^2y}{dx^2} = \frac{b}{a} \operatorname{cosec}^2 t \cdot \frac{1}{-a \sin t} = -\frac{b}{a^2} \operatorname{cosec}^3 t$$

64. $f(x) = x^3 - 3x^2 + 3x + 30$,

$$f'(x) = 3x^2 - 6x + 3$$

$$f''(x) = 6x - 6 \text{ and } f''(-3) = -24$$

65. $y = x^3 + \log x$.

$$y' = 3x^2 + \frac{1}{x}$$

$$y'' = 6x - \frac{1}{x^2}$$

66. $y = \tan^{-1} x$,

$$y_1 = \frac{1}{1+x^2} \Rightarrow (1+x^2)y_1 = 1$$

$$\Rightarrow (1+x^2)y_2 + 2xy_1 = 0$$

67. If $y = \sin^{-1} x$,

$$y_1 = \frac{1}{\sqrt{1-x^2}} \Rightarrow (1-x^2)y_1^2 - x^2y_1^2 = 1$$

$$\Rightarrow (1-x^2)y_2 - xy_1 = 0.$$

SAMPLE PAPER II

- Let N be the set of natural numbers and R be the relation in $N \times N$ defined by $(a,b) R (c,d)$ if $ad = bc$. Show that R is an equivalence relation.
- Prove that $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \frac{1}{2} \cos^{-1}\left(\frac{3}{5}\right)$

OR

Prove that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$.

- Using properties of determinants, show that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc.$$

- If $y = \sin^{-1}(\log x)$ prove that $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$.

5. For what values of a and b , the function 'f' defined by

$$f(x) = \begin{cases} 3ax+b & \text{if } x < 1 \\ 11 & \text{if } x = 1 \\ 5ax-2b & \text{if } x > 1 \end{cases} \text{ is continuous at } x=1.$$

6. Using the properties of definite integral, evaluate $\int_0^{\pi} \frac{x}{4-\cos^2 x} dx$.

7. Evaluate: $\int \frac{x}{x^3-1} dx$

8. Find a unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ & $\vec{a} - \vec{b}$ where

$$\vec{a} = 3\vec{i} + 2\vec{j} + 2\vec{k} \text{ \& } \vec{b} = \vec{i} + 2\vec{j} - 2\vec{k}$$

9. Find the particular solution of the differential equation: $(1+y+x^2y)dx + (x+x^3)dy = 0$ where $y = 0$ and $x = 1$.

10. A biased die is twice as likely to show an even number as an odd number. The die is rolled three times. If occurrence of an even number is considered a success, then write the probability distribution of number of success. Also find the mean number of success.

11. Solve the following differential equation $2x^2 dy/dx - 2xy + y^2 = 0$.

12. Draw a rough sketch of the region enclosed between the circles: $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 1$, using integration, find the area of the enclosed region.

13. In an examination 10 questions of true/false type are asked. A

student tosses a fair coin to determine his answer to each question. If the coin shows head, he answers 'true' and if it shows tail, he answers 'false'. Show that the probability that he answers at most 7 questions correctly is $\frac{121}{128}$.

OR

From a pack of 52 cards, a card is lost. From the remaining 51 cards, 2 cards are drawn at random (without replacement) are found to be both diamonds. What is the probability that the lost card was a card of heart?

14. A company sells two different products A and B . The two products are produced in a common production unit which has a total capacity of 500 hours. It takes 5 hours to produce a unit of A and 3 hours to produce a unit of B . The demand in the market shows the maximum number of units of A that can be sold is 70 and that of B is 120. Profit by selling one unit of A is Rs.20 and that of B is Rs.15. How many units of A and B should be produced to maximize the profit. Form L.P.P and solve it graphically.

MODEL SOLUTION

1. a) proving R is reflexive (1)
- b) proving R is symmetric (1)
- c) Proving R is transitive (1)
- d) Proving R is equivalence relation (1)

2. L.H.S ;

$$\begin{aligned} \text{a) } & \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) \\ &= \tan^{-1}\left(\frac{1}{2}\right) \quad (2) \end{aligned}$$

$$\text{b) } \frac{1}{2} \left\{ 2 \tan^{-1}\left(\frac{1}{2}\right) \right\} = \frac{1}{2} \cos^{-1}\left(\frac{1}{3}\right) \quad (2)$$

OR

$$\begin{aligned} \text{L.H.S} &= \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 \\ &= \frac{\pi}{4} + \left(\frac{\pi}{2} - \cot^{-1}2\right) + \left(\frac{\pi}{2} - \cot^{-1}3\right) \quad (1) \end{aligned}$$

$$= \frac{5\pi}{4} - \cot^{-1}2 - \cot^{-1}3$$

$$= \frac{5\pi}{4} - \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3} \quad (1)$$

$$= \frac{5\pi}{4} - \frac{\pi}{4} = \pi \quad (2)$$

3. Operate $R_1 \rightarrow R_1 - R_2 - R_3$ (1)
Expanding the determinant by R_1 (1)
get the result after simplification (2)
result = 4abc

4. We have $y = \sin(\log x)$
Diff. w.r.t x ,
 $x \frac{dy}{dx} = \cos(\log x)$ (1)

Again diff. w.r.t x $x \frac{d^2y}{dx^2} + \frac{dy}{dx} = -\sin(\log x) \cdot \frac{1}{x}$ $\left(1 \frac{1}{2}\right)$

Hence $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0 \left(1 \frac{1}{2}\right)$

5. LHL = $\lim_{x \rightarrow 1^-} f(x) = 5a - 2b$ $\left(\frac{1}{2}\right)$
RHL = $\lim_{x \rightarrow 1^+} f(x) = 3a + b$ $\left(\frac{1}{2}\right)$
 $x \rightarrow 1^+$

f is continuous at $x = 1$
 $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$
 $x = 1^+$
From this we get
 $5a - 2b = 1$

$3a + b = 1$ (1)

Solving above equations $a = 3, b = 2$ $b=21$)

6. $I = \int_0^\pi \frac{x}{4 - \cos^2 x} dx$

$I = \int_0^\pi \frac{\pi - x}{4 - \cos^2 x} dx$ [$\because \int_0^a f(x) dx = \int_0^a f(a-x) dx$] (1)

Now $2I = 2\pi \int_0^{\pi/2} \frac{dx}{4 - \cos^2 x} [\because \cos^2(\pi - x) = \cos^2 x]$

$= 2\pi \int_0^{\pi/2} \frac{\sec^2 x}{3 + 4 \tan^2 x} dx$ (1)

Put $t = \tan x$, $\sec^2 x dx = dt$, $x = \frac{\pi}{2}$,

After solution

$$2I = \frac{\pi}{\sqrt{3}}(\tan^{-1} \infty - \tan^{-1} 0) \quad (1)$$

$$\text{Hence } I = \frac{\pi^2}{4\sqrt{3}} (\text{since } \tan^{-1} \infty = \frac{\pi}{2}) \quad (1)$$

7. Let $I = \int \frac{x}{x^3-1} dx$

$$\frac{x}{x^3-1} = \frac{x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\text{Solving } A = C = \frac{1}{3}, B = -\frac{1}{3} \quad (1)$$

$$\text{Then } I = \frac{1}{3} \int \frac{1}{x-1} dx - \frac{1}{3} \int \frac{\frac{1}{2}(2x+1) - \frac{3}{2}}{x^2+x+1} dx \quad (1)$$

Integrating and solving

$$I = \frac{1}{3} \log |x-1| - \frac{1}{6} \log |x^2+x+1| + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + c \quad (2)$$

8. $(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c})$

$$(\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) - (\vec{d} \times \vec{b}) + (\vec{d} \times \vec{c}) \quad (1)$$

$$(\vec{c} \times \vec{d}) - (\vec{b} \times \vec{d}) + (\vec{b} \times \vec{c}) - (\vec{c} \times \vec{d}) \quad (1)$$

$$= 0 \quad [\because \vec{a} \times \vec{b} = \vec{c} \times \vec{d}, \vec{a} \times \vec{c} = \vec{b} \times \vec{d}] \quad (1)$$

$$\text{hence } (\vec{a} - \vec{d}) \text{ parallel } (\vec{b} - \vec{c}) \quad (1)$$

9. Given differential equation is

$$(1+y+x^2y)dx + (x+x^3)dy = 0$$

$$\frac{dy}{dx} = \frac{-1-y(1+x^2)}{x(1+x^2)} \quad (1/2)$$

$$\therefore \frac{dy}{dx} + \frac{1}{x}y = -\frac{1}{x(1+x^2)}$$

$$\text{Comparing } \frac{dy}{dx} + py = Q \quad (1/2)$$

$$\text{Finding I.F} = e^{\int \frac{1}{x} dx} = x$$

$$\frac{d}{dx}(y \cdot x) = -\frac{1}{1+x^2} \quad (1)$$

$$yx = -\tan^{-1}x + c \quad (1)$$

$$\text{Putting } y = 0, x = 1$$

$$yx = -\tan^{-1}x + \frac{\pi}{4} \text{ this is the result} \quad (1)$$

$$10. \text{ We have } P(\text{odd number}) = \frac{1}{3}$$

$$P(\text{even number}) = \frac{2}{3}$$

Let X be the event getting an even number
 ie. 0, 1, 2, 3 (1)

$$P(0) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

$$P(1) = \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{6}{27}$$

$$P(2) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{12}{27}$$

$$P(3) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27} \quad (1)$$

X	0	1	2	3
P(X)	$\frac{1}{27}$	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{8}{27}$
XP(X)	0	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{12}{27}$

$$\text{Mean} = \sum X P(X) = 2 \quad (2)$$

11. To convert into linear in y (1)

Correct I.F (1)

Correct solution (2)

12. Given circle are $x^2 + y^2 = 4$ -----(i)

$(x-2)^2 + y^2 = 1$ ----- (ii)

Equation (i) centre (0,0), radius = 2

Equation (ii) centre (2,0), radius = 1 (1)

For fig. (1)

Solving (i), (ii), $x = \frac{7}{3}$(1)

$$\text{Required area} = 2 \int_{1\sqrt{4}}^{\frac{7}{4}\sqrt{4}} \sqrt{1-(x-2)^2} dx + \int_{\frac{7}{4}\sqrt{4}}^2 \sqrt{4-x^2} dx \quad (1)$$

Solving

$$\text{Required area} = \frac{5\pi}{2} - \frac{\sqrt{15}}{2} - \sin^{-1}\left(\frac{1}{4}\right) - 4\sin^{-1}\left(\frac{7}{8}\right) \text{ sq. unit} \quad (2)$$

$$13. P(\text{answer is true}) = \frac{1}{2} \quad (p)$$

$$P(\text{answer is false}) = \frac{1}{2} \quad (q)$$

$$x = 100 \quad (1)$$

P(most 7 correct answer)

$$= 1 - P(8) + P(9) + p(10) \quad (2)$$

$$= 1 - \left(10_{c_2} + 10_{c_1} + 10_{c_0}\right) \left(\frac{1}{2}\right)^{10} \quad (1)$$

$$\text{Simplify} = \frac{121}{128} \text{ which is true} \quad (2)$$

OR

Let the events be

$E_1 \rightarrow$ missing card is a diamond

$E_2 \rightarrow$ missing card is a spade

$E_3 \rightarrow$ missing card is a club

$E_4 \rightarrow$ missing card is a heart

And $A \rightarrow$ 2 diamonds cards are drawn

(1)

$$P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4} \quad (1)$$

$$P\left(\frac{A}{E_1}\right) = \frac{12}{51} \times \frac{11}{50}$$

$$P\left(\frac{A}{E_2}\right) = P\left(\frac{A}{E_3}\right) = P\left(\frac{A}{E_4}\right) = \frac{13}{51} \times \frac{12}{50} \quad (1)$$

By Baye's theorem

$$P\left(\frac{E_4}{A}\right) = \frac{P(E_4) \cdot P\left(\frac{A}{E_4}\right)}{\sum_{i=1}^4 P(E_i) \cdot P\left(\frac{A}{E_i}\right)} \quad (1)$$

$$\text{After simplification } P\left(\frac{E_4}{A}\right) = \frac{13}{50} \quad (2)$$

14. Unit A produced = x

Unit B produced = y

The LPP : max. : $Z = 20x + 15y$ ---(i)

Subject to $5x + 3y \leq 500$ -----(ii)

$x \leq 70$ -----(iii)

$y \leq 120$, $x, y \geq 0$ -----(iv) (2)

For graph (2)

Conner points are

O(0,0), D(70,0), E(70,50), F(28,120), C(0,120) (1)

Maximum profit is Rs.2360 when (28,120) (1)

Topic : Vector

68. If $\vec{a} = 5\vec{i} - \vec{j} - 3\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j} - 5\vec{k}$ then show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular.

69. If $\vec{a} = 2\vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = -\vec{i} + 2\vec{j} + \vec{k}$ and $\vec{c} = 3\vec{i} + \vec{j}$ are such that $\vec{a} + \mu\vec{b}$ is perpendicular to \vec{c} , then find the value of μ .

70. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.

71. Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = 3\vec{i} + 2\vec{j} + 2\vec{k}$ and $\vec{b} = \vec{i} + 2\vec{j} - 2\vec{k}$.

Model Answers:

68. $\vec{a} + \vec{b} = 6\vec{i} + 2\vec{j} - 8\vec{k}$, $\vec{a} - \vec{b} = 4\vec{i} - 4\vec{j} + 2\vec{k}$, $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 24 - 8 - 16 = 0$
therefor $\vec{a} + \vec{b}$ is perpendicular to $\vec{a} - \vec{b}$.

69. $\vec{a} + \mu\vec{b} = (2\vec{i} + 2\vec{j} + 3\vec{k}) + \mu(-\vec{i} + 2\vec{j} + \vec{k}) = (2 - \mu)\vec{i} + (2 + 2\mu)\vec{j} + (3 + \mu)\vec{k}$
 $\vec{a} + \mu\vec{b}$ is perpendicular to \vec{c} , $\therefore (\vec{a} + \mu\vec{b}) \cdot \vec{c} = 0$, $\therefore 3(2 - \mu) + (2 + 2\mu) = 0$, $\therefore \mu = 8$.

70. $\vec{a}, \vec{b}, \vec{c}$ are unit vectors $\therefore |\vec{a}| = |\vec{b}| = |\vec{c}| = 1$, $\vec{a} \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \cdot \vec{0} = 0$,
 $|\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = 0$, $\therefore \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = -1$, similarly, $\vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{c} = -1$ and
 $\vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{a} = -1$, Add all above three equation, we get $2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = -3$,
 $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = \frac{-3}{2}$

71. $\vec{a} + \vec{b} = 4\vec{i} + 4\vec{j}$ and $\vec{a} - \vec{b} = 2\vec{i} + 4\vec{k}$, $\vec{c} =$

DIFFERENTIAL EQUATION

72. Form a Differential equation of the family of circles in the second quadrant and touching the coordinate axes.

73. Form a Differential equation representing the family of curves $y^2 y'' - 2ay + x^2 = a^2$, where a is an arbitrary constant.

74. Form a Differential equation representing the family of curves given by

$$(x-a)^2 + 2y^2 = a^2 \text{ where } a \text{ is an arbitrary constant.}$$

75. Form a Differential equation representing the family of circles touching the Y-axis at origin.

76. Form a Differential equation representing the family of circles touching the X-axis at origin.

SOLUTION

72. Equation representing the family of circles is $(x+a)^2 + (y-a)^2 = a^2$

$$x^2 + y^2 + 2ax - 2ay + a^2 = 0$$

Diff. the eq with r. to x we get

$$2x + 2y \frac{dy}{dx} + 2a - 2a \frac{dy}{dx} = 0$$

$$a = x + y \frac{y'}{y' - 1}$$

substituting the value in eq.

$$(x+y)^2 \left[\left(\frac{y'}{y'-1} \right)^2 + 1 \right] = (x+y \frac{y'}{y'-1})^2 \text{ is the required diff. eq.}$$

73. family of curves $y^2 - 2ay + x^2 = a^2$ diff. w.r to x

$$a = (y + xy') \text{ sub. Value of } a \text{ in eq}$$

$$p^2(x^2 - 2y^2) - 4xyp - x^2 = 0 \text{ where } p = \frac{dy}{dx}$$

74. Given family of curves $(x-a)^2 + 2y^2 = a^2$

$$2a = (x^2 + 2y^2)/x \text{ diff. both side w.r.to x we get}$$

$$(x^2 - 2y^2) = -4xy \frac{dy}{dx}$$

$$\frac{dy}{dx} = (x^2 - 2y^2)/-4xy \text{ is the required eq.}$$

75. Let, (r,0) be the centre of circle eq of circle will be $(x-r)^2 + (y-0)^2 = r^2$

$$\text{Diff w.r to } x, r = x + y \frac{dy}{dx}$$

Putting the value of r in the given eq. we get

$2xydy/dx + x^2 - y^2 = 0$ which is the required solution.

76. Let $(0, a)$ be the coordinates of the centre of any member of the family of circle the eq of family is $x^2 + (y-a)^2 = a^2$ or $x^2 + y^2 = 2ay$

Diff. w r to x we get $a = (x + ydy/dx)/dy/dx$

Sub. The value of a in the eq. we get $dy/dx = 2xy/x^2 - y^2$ this is the required diff. eq.

Questions based on variable separable

Q.1 Solve $\frac{dy}{dx} = \sec y$

Q2. Solve. $\frac{dy}{dx} = (1+x^2)(1+y^2)$

Q3. Solve $e^{\frac{dy}{dx}} = 2$

Q4. Solve $\frac{dy}{dx} = x^2 + \sin 3x$

Q5. Solve $xy \frac{dy}{dx} = (x+2)(y+2)ydy/dx = (x+2)(y+2)$

Solutions:-

Q1. $\frac{dy}{dx} = \sec y$

$$\frac{dy}{\sec y} = dx$$

$$\int \frac{dy}{\sec y} = \int dx$$

$\sin y = x + c$ Ans.

Q2. $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

$$\frac{dy}{1+y^2} = (1+x^2)dx$$

$$\int \frac{dy}{(1+y^2)} = \int (1+x^2)dx \quad \text{ } \int \frac{dy}{(1+y^2)} = \int 1+x^2 dx$$

$$\tan^{-1} y = x + \frac{x^3}{3} + c \quad \text{Ans.}$$

Q3. $e^{dy/dx} = 2$

$$\frac{dy}{dx} = \log 2 \quad dy/dx = \log 2$$

$$dy = \log 2 \, dx$$

$$\int dy = \int \log 2 \, dx \quad \text{ } \int dy = \int \log 2 \, dx$$

$$y = \log 2 \cdot x + c + c \quad \text{ans.}$$

Q4. $\frac{dy}{dx} = x^2 + \sin 3x$

$$dy = (x^2 + \sin 3x)dx$$

$$\int dy = \int (x^2 + \sin 3x)dx$$

$$y = \frac{x^3}{3} - \frac{\cos 3x}{3} + c \quad y = x^3/3 - \cos 3x/3 + c \quad \text{Ans.}$$

Q5. $\frac{xydy}{dx} = (x + 2)(y + 2)$

$$\frac{ydy}{y+2} = \frac{x+2}{x}dx$$

$$\int \frac{y dy}{y+2} = \int \frac{x+2}{x} dx$$

$$y - 2 \log(y+2) = x + 2 \log x + c$$

VECTORS

- Q-1 Show that the vectors $\vec{a} = 3\vec{i} - 2\vec{j} - 5\vec{k}$ and $\vec{b} = 6\vec{i} - \vec{j} + 4\vec{k}$ are orthogonal.
- Q-2 Show that the vectors $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and $\vec{b} = 4\vec{i} + 6\vec{j} + 2\vec{k}$ are orthogonal.
- Q-3 Find the value of 't' so that the vectors $\vec{a} = 2\vec{i} + t\vec{j} + \vec{k}$ and $\vec{b} = \vec{i} - 2\vec{j} + 3\vec{k}$ are orthogonal.
- Q-4 Find the value of 't' so that the vectors $\vec{a} = 3\vec{i} + t\vec{j} + \vec{k}$ and $\vec{b} = 2\vec{i} - \vec{j} - 8\vec{k}$ are orthogonal.
- Q-5 If $\vec{\alpha} = \vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{\beta} = 3\vec{i} - \vec{j} - 2\vec{k}$, find $(2\vec{\alpha} + \vec{\beta}) \cdot (\vec{\alpha} + 2\vec{\beta})$.
- Q-6 If $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$, $\vec{b} = 5\vec{i} + 4\vec{j} - \vec{k}$, $\vec{c} = 3\vec{i} + 6\vec{j} + 2\vec{k}$ and $\vec{d} = \vec{i} + 2\vec{j}$, show that $\vec{b} - \vec{a}$ is perpendicular to $\vec{d} - \vec{c}$.
- Q-7 Show that the vectors $\vec{a} = 2\vec{i} + 3\vec{j} - 6\vec{k}$, $\vec{b} = 3\vec{i} - 6\vec{j} - 2\vec{k}$, $\vec{c} = 6\vec{i} + 2\vec{j} + 3\vec{k}$ are mutually perpendicular.
- Q-8 verify if the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular or not if it is given that $|\vec{a}| = |\vec{b}|$.
- Q-9 If $\vec{\alpha} = \vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{\beta} = 3\vec{i} - \vec{j} - 2\vec{k}$, Find $(\vec{\alpha} + \vec{\beta}) \cdot (\vec{\alpha} - \vec{\beta})$, hence deduce that the vectors $\vec{\alpha} + \vec{\beta}$ and $\vec{\alpha} - \vec{\beta}$ are mutually perpendicular.
- Q-10 Find the angle between the vectors $\vec{a} = 3\vec{i} + 2\vec{j} + \vec{k}$ and $\vec{b} = 2\vec{i} - \vec{j} - 4\vec{k}$

using the scalar product of the vectors.

SOLUTIONS

Q-1 Here $\vec{a} \cdot \vec{b} = 18+2-20 = 0$ hence $\vec{a} \cdot \vec{b} = 0$ so the vectors are orthogonal.

Q-2 Here $\vec{a} \cdot \vec{b} = 8-18+10 = 0$ hence $\vec{a} \cdot \vec{b} = 0$ so the vectors are orthogonal.

Q-3 We know that two vectors \vec{a} and \vec{b} are orthogonal if $\vec{a} \cdot \vec{b} = 0$

Here $\vec{a} = 2\vec{i} + t\vec{j} + \vec{k}$ and $\vec{b} = \vec{i} - 2\vec{j} + 3\vec{k}$ are orthogonal.

$$\text{So, } \vec{a} \cdot \vec{b} = 0$$

$$\text{So, } 2 - 2t + 3 = 0$$

$$\text{Hence } t = \frac{5}{2}$$

Q-4 We know that two vectors \vec{a} and \vec{b} are orthogonal if $\vec{a} \cdot \vec{b} = 0$

Here $\vec{a} = 3\vec{i} + t\vec{j} + \vec{k}$ and $\vec{b} = 2\vec{i} - \vec{j} - 8\vec{k}$ are orthogonal.

$$\text{So, } \vec{a} \cdot \vec{b} = 0$$

$$\text{So, } 6 - t - 8 = 0$$

$$\text{Hence } t = -2$$

Q-5 Here $\vec{\alpha} = \vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{\beta} = 3\vec{i} - \vec{j} - 2\vec{k}$

$$\Rightarrow 2\vec{\alpha} + \vec{\beta} = (2\vec{i} + 4\vec{j} - 6\vec{k}) + (3\vec{i} - \vec{j} - 2\vec{k}) = 5\vec{i} + 3\vec{j} - 8\vec{k}$$

$$\text{And } \vec{\alpha} + 2\vec{\beta} = (\vec{i} + 2\vec{j} - 3\vec{k}) + (6\vec{i} - 2\vec{j} - 4\vec{k}) = 7\vec{i} - 7\vec{k}$$

$$\Rightarrow (2\vec{\alpha} + \vec{\beta}) \cdot (\vec{\alpha} + 2\vec{\beta}) = (5\vec{i} + 3\vec{j} - 8\vec{k}) \cdot (7\vec{i} - 7\vec{k}) = 35 + 56 = 91$$

Q-6 Here $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$, $\vec{b} = 5\vec{i} + 4\vec{j} - \vec{k}$, $\vec{c} = 3\vec{i} + 6\vec{j} + 2\vec{k}$ and $\vec{d} = \vec{i} + 2\vec{j}$

$$\Rightarrow \vec{b} - \vec{a} = (5\hat{i} + 4\hat{j} - \hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k}) = 3\hat{i} + \hat{j} - 5\hat{k} \text{ and}$$

$$\vec{d} - \vec{c} = (\hat{i} + 2\hat{j}) - (3\hat{i} + 6\hat{j} + 2\hat{k}) = -2\hat{i} - 4\hat{j} - 2\hat{k}$$

$$\Rightarrow (\vec{b} - \vec{a}) \cdot (\vec{d} - \vec{c}) = (3\hat{i} + \hat{j} - 5\hat{k}) \cdot (-2\hat{i} - 4\hat{j} - 2\hat{k}) = -6 - 4 + 10 = 0$$

$$\Rightarrow \vec{b} - \vec{a} \text{ is perpendicular to } \vec{d} - \vec{c}$$

Q-7 Here $\vec{a} = 2\hat{i} + 3\hat{j} - 6\hat{k}$, $\vec{b} = 3\hat{i} - 6\hat{j} - 2\hat{k}$, $\vec{c} = 6\hat{i} + 2\hat{j} + 3\hat{k}$.

$$\vec{a} \cdot \vec{b} = (2\hat{i} + 3\hat{j} - 6\hat{k}) \cdot (3\hat{i} - 6\hat{j} - 2\hat{k}) = 6 - 18 + 12 = 0$$

$$\vec{b} \cdot \vec{c} = (3\hat{i} - 6\hat{j} - 2\hat{k}) \cdot (6\hat{i} + 2\hat{j} + 3\hat{k}) = 18 - 12 - 6 = 0$$

$$\vec{c} \cdot \vec{a} = (6\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (2\hat{i} + 3\hat{j} - 6\hat{k}) = 12 + 6 - 18 = 0$$

$$\Rightarrow \text{vectors } \vec{a}, \vec{b} \text{ and } \vec{c} \text{ are mutually perpendicular.}$$

Q-8 $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b}$

$$= |\vec{a}|^2 - |\vec{b}|^2 \text{ (because dot product is commutative)}$$

$$= 0 \quad (\text{because } |\vec{a}| = |\vec{b}| \text{ is given})$$

Q-9 Here $\vec{\alpha} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{\beta} = 3\hat{i} - \hat{j} - 2\hat{k}$

$$\Rightarrow \vec{\alpha} + \vec{\beta} = (\hat{i} + 2\hat{j} - 3\hat{k}) + (3\hat{i} - \hat{j} - 2\hat{k}) = 4\hat{i} + \hat{j} - 5\hat{k} \text{ and}$$

$$\vec{\alpha} - \vec{\beta} = (\hat{i} + 2\hat{j} - 3\hat{k}) - (3\hat{i} - \hat{j} - 2\hat{k}) = -2\hat{i} + 3\hat{j} - \hat{k}$$

$$\Rightarrow (\vec{\alpha} + \vec{\beta}) \cdot (\vec{\alpha} - \vec{\beta}) = (4\hat{i} + \hat{j} - 5\hat{k}) \cdot (-2\hat{i} + 3\hat{j} - \hat{k}) = -8 + 3 + 5 = 0$$

$$\text{Here } (\vec{\alpha} + \vec{\beta}) \cdot (\vec{\alpha} - \vec{\beta}) = 0 \text{ hence } \vec{\alpha} + \vec{\beta} \text{ and } \vec{\alpha} - \vec{\beta} \text{ are mutually perpendicular.}$$

Q-10 Here $\vec{a} = 3\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} - 4\hat{k}$

$$\Rightarrow \vec{a} \cdot \vec{b} = (3\hat{i} + 2\hat{j} + \hat{k}) \cdot (2\hat{i} - \hat{j} - 4\hat{k}) = 6 - 2 - 4 = 0$$

And according to the definition of the scalar product we know that if the scalar product of two vectors is zero then the vectors are mutually perpendicular.

Here $\vec{a} \cdot \vec{b} = 0$ hence the angle between the vectors is $\frac{\pi}{2}$.

VECTORS

1. If $\vec{a} = i - 2j + 3k$ and $\vec{b} = 2i + 3j - 5k$, then find $\vec{a} \times \vec{b}$ and verify $\vec{a} \times \vec{b}$ is perpendicular to \vec{a}

$$\text{Ans } \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 2 & 3 & -5 \end{vmatrix} = i + 11j + 7k$$

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = (i + 11j + 7k) \cdot (i - 2j + 3k) = 0$$

2. If vectors $\vec{a} = 2i + 2j + 3k$, $\vec{b} = -i + 2j + k$ and $\vec{c} = 3i + j$ are such that $\vec{a} + \mu \vec{b}$ is perpendicular to \vec{c} , then find the value of μ .

$$\text{Ans. } (\vec{a} + \mu \vec{b}) \cdot \vec{c} = 0$$

Hence $\mu = 8$.

3. Find the angle between the vectors $2i - 3j + 6k$ and $2i - 3j - 5k$

$$\text{Ans } \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{-17}{7\sqrt{38}}$$

$$\text{Hence } \theta = \cos^{-1} \left(\frac{-17}{7\sqrt{38}} \right)$$

4. Find the value of x for which the vector $x(i + j + k)$ is a unit vector.

$$\text{Ans if it is a unit vector then } |xi + xj + xk| = 1$$

$$\sqrt{x^2 + x^2 + x^2} = 1$$

$$\text{So } x = \pm \frac{1}{\sqrt{3}}$$

5. Find the unit vector perpendicular to two vectors $\vec{a} = i - 2j + 3k$ and $\vec{b} = 2i + 3j - 5k$

$$\text{Ans } \vec{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & -2 & 3 \\ 2 & 3 & -5 \end{vmatrix} = i + 11j + 7k$$

$$|\vec{a} \times \vec{b}| = \sqrt{1^2 + 11^2 + 7^2} = \sqrt{171}$$

$$\text{So } \vec{n} = \frac{i + 11j + 7k}{\sqrt{171}}$$

6. For what value of μ , are the vectors $2i + \mu j + k$ and $i - 2j + 3k$ perpendicular to each other.

$$\text{Ans } (2i + \mu j + k) \cdot (i - 2j + 3k) = 0$$

$$2 - 2\mu + 3 = 0 \quad \text{so } \mu = \frac{5}{2}$$

7. Find a unit vector in the direction of sum of the vectors $\vec{a} = i - 2j + 7k$ and $\vec{b} = 2i + 2j - 3k$.

$$\text{Ans } \vec{c} = \vec{a} + \vec{b} = 3i + 4k$$

$$\text{Hence } \vec{n} = \frac{\vec{c}}{|\vec{c}|} = \frac{3i + 4k}{5}$$

8. If $|\vec{a}| = 2$, $|\vec{b}| = 5$ and angle between them is 60° find $|\vec{a} \times \vec{b}|$

$$\text{Ans } |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin 60^\circ \\ = 2 \times 5 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}$$

9. Evaluate $i \times (j \times k) + j \cdot (i \times k) - k \cdot (i \times j)$

$$\text{Ans } i \times i + j \times (-j) - k \cdot 0 = 0 - j^2 - 0 = -1$$

10. For what value of α , vectors $i + j - 2k$, $i - 3j + 4k$ and $2i + j + \alpha k$

are coplanar.

$$\text{Ans } \begin{vmatrix} 1 & 1 & -2 \\ 1 & -3 & 4 \\ 2 & 1 & \alpha \end{vmatrix} = 0$$

$$-3\alpha - 4 - \alpha + 8 - 14 = 0$$

$$\alpha = \frac{-5}{2}$$

11. Find a vector whose magnitude is 3 units and which is perpendicular to the vectors \vec{a} and \vec{b} where $\vec{a} = (3\hat{i} + \hat{j} - 4\hat{k})$, $\vec{b} = (6\hat{i} + 5\hat{j} - 2\hat{k})$

$$\text{Ans: } (2\hat{i} - 2\hat{j} + \hat{k}) \text{ or } -(2\hat{i} - 2\hat{j} + \hat{k})$$

$$\text{Let } \vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}, \vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k} \text{ and } \vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}.$$

Find a vector \vec{d} which is perpendicular to both \vec{a} and \vec{b} and $\vec{c} \cdot \vec{d} = 18$.

$$\text{Sol. Let } \vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{We have } \vec{a} \cdot \vec{d} = 0, \vec{b} \cdot \vec{d} = 0, \vec{c} \cdot \vec{d} = 18.$$

Which gives

$$x + 4y + 2z = 0$$

$$3x - 2y + 7z = 0$$

$$2x - y + 4z = 18$$

By solving these equations we get

x, y and z then we find \vec{d}

12. If \vec{a}, \vec{b} and \vec{c} are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.

Hint- Take dot product of $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ with \vec{a}, \vec{b} and \vec{c} and solve.

$$\text{Answer } \frac{-3}{2}$$

13. Find a unit vector perpendicular to each of the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ where $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{b} = \vec{i} + 2\vec{j} + 3\vec{k}$.

Hint- Evaluate $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$. get the cross product of these vectors and divide it by its modulus.

Answer $\frac{1}{\sqrt{24}}(-2\vec{i} + 4\vec{j} - 2\vec{k})$

14. If $\vec{a} = 4\vec{i} + 5\vec{j} - \vec{k}$; $\vec{b} = \vec{i} - 4\vec{j} + 5\vec{k}$ and $\vec{c} = 3\vec{i} + 5\vec{j} - \vec{k}$, find a vector \vec{d} which is perpendicular to both \vec{a} and \vec{b} and satisfying $\vec{c} \cdot \vec{d} = 21$.

15. Let $\vec{a} = \vec{i} + 4\vec{j} + 2\vec{k}$, $\vec{b} = 3\vec{i} - 2\vec{j} + 7\vec{k}$ and $\vec{c} = 2\vec{i} - \vec{j} + 4\vec{k}$. Find a vector \vec{d} which is perpendicular to both the vectors \vec{a} and \vec{b} , and $\vec{c} \cdot \vec{d} = 15$.

Answer $\frac{1}{3}(160\vec{i} - 5\vec{j} + 70\vec{k})$

16. If vectors $\vec{a}, \vec{b}, \vec{c}$ satisfy the condition $(\vec{a} + \vec{b} + \vec{c}) = 0$ and $|\vec{a}| = 1$, $|\vec{b}| = 4$, $|\vec{c}| = 2$ then find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$

Sol: We have

$$(\vec{a} + \vec{b} + \vec{c})^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a} = 1^2 + 4^2 + 2^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$$

Since $(\vec{a} + \vec{b} + \vec{c}) = 0$, we have

$$0 = 21 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) \Rightarrow \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = \frac{-21}{2} \Rightarrow \mu = \frac{-21}{2}$$

THREE DIMENSIONAL GEOMETRY

SHORTEST DISTANCE

Q1. Find the shortest distance between the lines whose vector equations

$$\text{are } \vec{r} = \begin{pmatrix} i + 2j + 3k \end{pmatrix} + \lambda \begin{pmatrix} i - 3j - 2k \end{pmatrix} \text{ and } \\ \vec{r} = \begin{pmatrix} 4i + 5j + 6k \end{pmatrix} + \mu \begin{pmatrix} 2i + 3j + k \end{pmatrix}$$

Sol. Here,

$$\vec{a}_1 = i + 2j + 3k, \vec{a}_2 = 4i + 5j + 6k$$

$$\vec{b}_1 = i - 3j + 2k, \vec{b}_2 = 2i + 3j + k$$

$$\text{So, } \vec{a}_2 - \vec{a}_1 = 3i + 3j + 3k$$

$$\text{And } \vec{b}_1 \times \vec{b}_2 = -a i + 3j + 9k$$

$$\text{Shortest distance } d = \frac{|\vec{(a_2 - a_1)} \cdot \vec{(b_1 \times b_2)}|}{|\vec{b_1 \times b_2}|}$$

$$= \frac{3}{\sqrt{19}} \text{ units.}$$

Q2. Find the shortest distance between the lines

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

Sol: The equations of the given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

It is known that the shortest distance between the lines,

$$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1 \text{ and } \vec{r} = \vec{a}_2 + \mu \vec{b}_2, \text{ is}$$

Given by,

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right| \quad \dots(1)$$

Comparing the given equations, we obtain

$$\begin{aligned} \vec{a}_1 &= \hat{i} + 2\hat{j} + \hat{k} \\ \vec{b}_1 &= \hat{i} - \hat{j} + \hat{k} \\ \vec{a}_2 &= 2\hat{i} - \hat{j} - \hat{k} \\ \vec{b}_2 &= 2\hat{i} + \hat{j} + 2\hat{k} \\ \vec{a}_2 - \vec{a}_1 &= (2\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 2\hat{k} \end{aligned}$$

$$\begin{aligned} \vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} \\ \vec{b}_1 \times \vec{b}_2 &= (-2-1)\hat{i} - (2-2)\hat{j} + (1+2)\hat{k} = -3\hat{i} + 3\hat{k} \\ \Rightarrow |\vec{b}_1 \times \vec{b}_2| &= \sqrt{(-3)^2 + (3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2} \end{aligned}$$

Substituting all the values in equation (1), we obtain

$$\begin{aligned} d &= \left| \frac{(-3\hat{i} + 3\hat{k}) \cdot (\hat{i} - 3\hat{j} - 2\hat{k})}{3\sqrt{2}} \right| \\ \Rightarrow d &= \left| \frac{-3.1 + 3(-2)}{3\sqrt{2}} \right| \\ \Rightarrow d &= \left| \frac{-9}{3\sqrt{2}} \right| \\ \Rightarrow d &= \frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2} \end{aligned}$$

Therefore, the shortest distance between the two lines is $\frac{3\sqrt{2}}{2}$ units.

Q3. Find the shortest distance between the lines:

$$\vec{r} = (\hat{i} + \hat{j} - k) + \lambda(3\hat{i} - \hat{j}) \text{ and } \vec{r} = (4\hat{i} - k) + \mu(2\hat{i} + 3k) .$$

Sol: Here

$$\vec{a}_1 = \vec{i} + \vec{j} - \vec{k}$$

$$\vec{b}_1 = 3\vec{i} - \vec{j}$$

$$\vec{a}_2 = 4\vec{i} - \vec{k}$$

$$\vec{b}_2 = 2\vec{i} + 3\vec{k}$$

So,

$$\vec{a}_2 - \vec{a}_1 = 3\vec{i} - \vec{j}$$

And

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = -3\vec{i} - 9\vec{j} + 2\vec{k}$$

$$\text{hence } |\vec{b}_1 \times \vec{b}_2| = \sqrt{94} .$$

Now we have the formula

$$\begin{aligned} \text{S.D} &= \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|} \\ &= (3\vec{i} - \vec{j}) \cdot (-3\vec{i} - 9\vec{j} + 2\vec{k}) / \sqrt{94} \\ &= 0 \end{aligned}$$

Q4. Find the shortest distance between the two lines

$$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k}) \text{ and } \vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k}) .$$

Ans: 9

Q5. Find the shortest distance between the lines

$$r = (i + 2j + 3k) + \lambda(i - 3j + 2k) \text{ and } r = (4i + 5j + 6k) + \lambda(2i + 3j + k).$$

Q6. Find the shortest distance between the lines

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \quad \text{and} \quad \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}.$$

EQUATION OF A PLANE

Q1. Find the equation of the plane passing through the points (0,-1,0), (1,1,1) and (3,3,0).

Sol: points (0,-1,0), (1,1,1), (3,3,0).

Equation of plane passing through point (0,-1,0)

$$A(x-0) + B(y+1) + C(z-0) = 0 \quad \dots\dots\dots(i)$$

Point (1, 1, 1) and (3,3,0) satisfy equ.(i) we get

$$A + 2B + C = 0 \quad \dots\dots\dots(ii)$$

$$3A + 4B + 0.C = 0 \quad \dots\dots\dots(iii)$$

Solve equ. (i) @ (ii)

$$\frac{A}{-4} = \frac{B}{3} = \frac{C}{-2} = \lambda \Rightarrow A$$

$$A = -4\lambda, B = 3\lambda, C = -2\lambda$$

Putting in (1)

$$-4\lambda x + 3\lambda(y+1) - 2\lambda z = 0$$

$$-4x + 3(y+1) - 2z = 0$$

$$4x + 3y + 2z - 3 = 0 \text{ Ans.}$$

Q2. Find the equation of the plane determined by the points (3,-1,2), (5,2,4) and (-1,-1,6).

Answer: $3x - 4y + 3z = 19$.

Q3. Find the equation of the plane passing through the points (2,1,0), (3,-2,-2) and (3,1,7).

Answer: $7x + 3y - z = 17$.

Q4. Find the equation of plane passing through the line of intersection of the planes $x + 2y + 3z = 4$ and $2x + y - z + 5 = 0$ and perpendicular to the plane $5x + 3y - 6z + 8 = 0$.

Sol: The required plane is $(x+2y + 3z) + k (2x + y - z +5) = 0$

Or $(1+2k)x + (2+k)y + (3-k)z -4 +5k = 0$:

Since it is perpendicular to $5x + 3y - 6z + 8 = 0$,

$5(1+2k) + 3(2+k) - 6(3-k) = 0$, i.e $k = 7/19$,

The equation of the plane is : $33x + 45y + 50z = 41$.

Q5. Find the equation of the plane through the intersection of the planes
 $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point $(2, 2, 1)$

Ans $7x - 5y + 4z - 8 = 0$

[Hint Use $(3x - y + 2z - 4) + k(x + y + z - 2) = 0$ and find k by using given point.]

Q6. Find the equation of the plane through the line of intersection of the
 planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the
 plane $x - y + z = 0$.

Ans: $x - z + 2 = 0$

Hint: Use $(x + y + z - 1) + k(2x + 3y + 4z - 5) = 0$ and find K using formula
 $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Q7. Find the Cartesian as well as vector equations of the planes, through
 the intersection of planes $\vec{r} \cdot (2\hat{i} + 6\hat{j}) + 12 = 0$ and $\vec{r} \cdot (3\hat{i} - \hat{j} + 4\hat{k}) = 0$
 which are at a unit distance from the origin.

Hint: convert into cartesian form.

Q8. Find the equation of the plane through the intersection of the planes
 $2x + y - 3z + 4 = 0$ and $3x + 4y + 8z - 1 = 0$ and making equal
 intercepts on the coordinate axes.

Ans: $5x + 5y + 5z = 3$

Q9. Find the equation of the plane passing through the line of

intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and
 parallel to x-axis.

Ans: $y - 3z + 6 = 0$

Q10. Find the equation of the plane which contains the line of

intersection of the planes $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$, $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$ and
 which is perpendicular to the plane $\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$

Ans: $33x+45y+50z = 41$

IMAGE OF A POINT IN A PLANE

Q1 Find the coordinates of the foot of the perpendicular drawn from origin to the plane $2x - 3y + 4z - 6 = 0$. Also find the image of the origin in plane.

Ans. Let L be the foot of the perpendicular from the origin to the plane

$$\text{Equation of OL is } \frac{x-0}{2} = \frac{y-0}{-3} = \frac{z-0}{4} = \mu$$

Therefore L $(2\mu, -3\mu, 4\mu)$

$$\text{SO } 2(2\mu) - 3(-3\mu) + 4(4\mu) - 6 = 0$$

$$\mu = \frac{6}{29}$$

Foot of the perpendicular L $(\frac{12}{29}, \frac{-18}{29}, \frac{24}{29})$

Let P (α, β, γ) be the image of O $(0, 0, 0)$

$$(\frac{\alpha+0}{2}, \frac{\beta+0}{2}, \frac{\gamma+0}{2}) = (\frac{12}{29}, \frac{-18}{29}, \frac{24}{29})$$

Therefore image is $(\frac{12}{29}, \frac{-18}{29}, \frac{24}{29})$

Q2. Find the co-ordinate of the foot of the perpendicular and distance of the point $(1, 3, 4)$ from the plane $2x - y + z + 3 = 0$. Find also the image of the point in the plane.

Sol. The given plane is $2x - y + z + 3 = 0$

Let P $(1, 3, 4)$ be a given point

M be the foot of perpendicular from P to the plane

Let $P'(\alpha, \beta, \gamma)$ be the image of P .

The equation of PM is

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} \dots\dots\dots (i)$$

Any point on eqn. (i) is

$$(1+2\lambda, 3-\lambda, 4+\lambda)$$

$$\text{If } 2(1+2\lambda) - (3-\lambda) + (4+\lambda) + 3 = 0$$

Finding $\lambda = -1$

Co-ordinate of M is $(-1, 4, 3)$

Perpendicular distance $|PM| = \sqrt{6}$ unit

Now M is the mid-point of PP'

If P' is (α, β, γ) then

$$\alpha = -3, \beta = 5, \gamma = 2$$

Hence P' image is $(-3, 5, 2)$

Q3. Find the image of the point $(1, 3, 4)$ in the plane $x - y + z = 5$.

Answer: $(3, 1, 6)$

Q4. Find the foot of perpendicular and image of the point $(1, 3, 4)$ in the plane $2x - y + z + 3 = 0$.

Q5. Find the foot of perpendicular and image of the point $P(1, 2, 4)$ in the plane $2x + y - 2z + 3 = 0$.

Questions related Bays Theorem

1. There are three coins. One is a two headed coin (having head on both faces), another is biased coin that comes up head 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows head, what is the probability that it was two headed coin?
2. In a bolt factory machines A, B and C manufacture respectively 25% , 35% and 40% of the total bolts of their output 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product. If the bolt drawn is found to be defective, what is the probability that it is manufactured by the machine B?

3. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter, a car and truck is 0.01 and 0.03 and 0.15 respectively. One of the insured person meets with an accident. What is the probability that he is a scooter driver?
4. An insurance company insured 2000 scooters and 3000 motor cycles. The probability of an accident involving a scooter is 0.01 and that of a motor cycle is 0.02. An insured vehicle met with an accident. Find the probability that the accidental vehicle was a motor cycle.
5. Two groups are competing for the position on the Board of directors of corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the product introduced by the second group. "Winners don't do different things. They do things differently." Point out the importance of innovative and creative thinking.
6. Given three identical boxes I, II and III , each containing two coins. In box I, both coins are gold coins, in box II ,both coins are silver coins and in box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold , what is the probability that the other coin in the box is also of gold ?
7. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually six.

8. In answering on a multiple choice test , a student either knows the answer or guesses . Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?
9. A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However the test also yields a false positive result for 0.5% of healthy person tested. If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive?
10. A card from a pack 52 cards is lost . From the remaining of the pack , two cards are drawn and are found to be both diamonds. Find the probability that actually there was a head.

Answers of questions related Bays Theorem

1. Let E_1 : Coin chosen is two headed

E_2 : Coin chosen is biased

E_3 : Coin chosen is un biased

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

A = Tossed coin shows head up

$$P(A/E_1) = 1, P(A/E_2) = \frac{75}{100} = \frac{3}{4} P(A/E_3) = \frac{1}{2}$$

$$P(E_1/A) = \frac{P(E_1) P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3) P(A/E_3)}$$

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times \frac{3}{4} + \frac{1}{3} \times \frac{1}{2}} = \frac{1}{9} = \frac{9}{4}$$

2. Let E_1 : the bolt is manufactured by machine A

E_2 : the bolt is manufactured by machine B

E_3 : the bolt is manufactured by machine C

$$P(E_1) = \frac{25}{100}, \quad P(E_2) = \frac{35}{100}, \quad P(E_3) = \frac{40}{100}$$

E = the bolt is defective

$$P(E/E_1) = \frac{5}{100}, P(E/E_2) = \frac{4}{100} P(E/E_3) = \frac{2}{100}$$

$$P(E_2/E) = \frac{P(E_2) P(E/E_2)}{P(E_1)P(E/E_1) + P(E_2)P(E/E_2) + P(E_3) P(E/E_3)}$$

$$\begin{aligned}
 &= \frac{\frac{35}{100} \times \frac{4}{100}}{\frac{25}{100} \times \frac{5}{100} + \frac{35}{100} \times \frac{4}{100} + \frac{40}{100} \times \frac{2}{100}} \\
 &= \frac{140}{125+140+80} = \frac{140}{345} = \frac{28}{69}
 \end{aligned}$$

3. Let

E_1 : the event that the insured person is a scooter driver

E_2 : the event that the insured person is a car driver

E_3 : the event that the insured person is a truck driver

$$P(E_1) = \frac{2000}{2000+4000+6000} = \frac{2000}{12000} = \frac{1}{6},$$

$$P(E_2) = \frac{4000}{12000} = \frac{1}{3} \quad \text{and} \quad P(E_3) = \frac{6000}{12000} = \frac{1}{2}$$

A = the event that the insured person meets an accident

$$P(A/E_1) = 0.01, P(A/E_2) = 0.03, P(A/E_3) = 0.15$$

Required probability

$$P(E_1/A) = \frac{P(E_1) P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3) P(A/E_3)}$$

$$= \frac{\frac{1}{6} \times 0.01}{\frac{1}{6} \times 0.01 + \frac{1}{3} \times 0.03 + \frac{1}{2} \times 0.15}$$

$$= \frac{0.01}{0.01 + 0.06 + 0.45} = \frac{0.01}{0.52} = \frac{1}{52}$$

4. Let

E_1 : the event that the scooter is insured

E_2 : the event that the motor cycle is insured

$$P(E_1) = \frac{2000}{2000+3000} = \frac{2}{5}, \quad P(E_2) = \frac{3000}{5000} = \frac{3}{5},$$

A = the event that the motor cycle met with an accident

$$P(A/E_1) = 0.01, \quad P(A/E_2) = 0.02$$

Required probability

$$\begin{aligned}
 P(E_2/A) &= \frac{P(E_2) P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)} \\
 &= \frac{\frac{3}{5} \times 0.02}{\frac{2}{5} \times 0.01 + \frac{3}{5} \times 0.02} \\
 &= \frac{0.06}{0.02 + 0.06} = \frac{0.06}{0.08} = \frac{6}{8} = \frac{3}{4}
 \end{aligned}$$

5. Let E_1 : the event that the I group wins

E_2 : the event that the II group wins

$$P(E_1) = \frac{6}{10}, \quad \text{and} \quad P(E_2) = \frac{4}{10}$$

A = the event that the new product is introduced

$$P(A/E_1) = \frac{7}{10}, P(A/E_2) = \frac{3}{10}$$

Required probability

$$\begin{aligned}
 P(E_2/A) &= \frac{P(E_2) P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)} \\
 &= \frac{\frac{4}{10} \times \frac{3}{10}}{\frac{6}{10} \times \frac{7}{10} + \frac{4}{10} \times \frac{3}{10}}
 \end{aligned}$$

$$= \frac{12}{42+12} = \frac{12}{54} = \frac{2}{9}$$

Rote learning must be avoided and learning by doing should be promoted.

6. Let E_1, E_2, E_3 be the events that boxes I, II, III are chosen respectively

$$P(E_1) = P(E_2) = P(E_3) = 1/3$$

Let A be the event that the coin is of gold

$$P(A/E_1) = \frac{2}{2}, P(A/E_2) = 0, P(A/E_3) = \frac{1}{2},$$

By Bay's theorem

$$\begin{aligned} P(E_1/A) &= \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + P(E_3)P\left(\frac{A}{E_3}\right)} \\ &= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2}} \\ &= \frac{2}{3} \end{aligned}$$

7. Let E_1, E_2 are the events that six occurs and six does not occur in throwing a die respectively.

$$P(E_1) = \frac{1}{6}, P(E_2) = \frac{5}{6}$$

Let A be the event that the man reports that six occurs on the die

$$P(A/E_1) = \frac{3}{4}, P(A/E_2) = \frac{1}{4},$$

By Bay's theorem

$$\begin{aligned} P(E_1/A) &= \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)} \\ &= \frac{3}{8} \end{aligned}$$

8. Let E_1, E_2 are the events that a student knows the answer and he guesses at the answer respectively.

$$P(E_1) = \frac{3}{4}, P(E_2) = \frac{1}{4}$$

Let A be the event that the man reports that six occurs on the die

$$P(A/E_1) = 1, P(A/E_2) = \frac{1}{4},$$

By Bay's theorem

$$\begin{aligned} P(E_1/A) &= \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)} \\ &= \frac{12}{13} \end{aligned}$$

9. Let E_1, E_2 are the events that person has the disease and person

is healthy respectively.

$$P(E_1) = 0.1\% = \frac{1}{100}, P(E_2) = 0.1\% = \frac{999}{1000}$$

Let A be the event that the man reports that six occurs on the die

$$P(A/E_1) = \frac{99}{100}, P(A/E_2) = 0.5\% = \frac{5}{1000},$$

By Bay's theorem

$$\begin{aligned} P(E_1/A) &= \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)} \\ &= \frac{198}{1197} \end{aligned}$$

10. Let E_1 , E_2 are the events that lost is diamond and lost card is not diamond respectively.

$$P(E_1) = \frac{13}{52} = \frac{1}{4}, P(E_2) = \frac{3}{4}$$

Let A be the event that the two cards drawn from the remaining cards are diamond

$$P(A/E_1) = \frac{c(12,2)}{c(51,2)} = \frac{12 \times 11}{51 \times 50}, P(A/E_2) = \frac{c(13,2)}{c(51,2)} = \frac{13 \times 12}{51 \times 50},$$

By Bay's theorem

$$P(E_1/A) = \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)} = \frac{11}{50}$$

QUESTIONS BASED ON Probability Distribution

- Find the probability distribution of number of successes in two tosses of a die, where a success is defined as
 - Number greater than 4
 - 6 appear on at least one die.
- A random variable X has the following probability distribution

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	K ²	2K ²	7K ² +k

Determine

- K
 - $P(X < 3)$
 - $P(X > 6)$;
 - $P(0 < X < 3)$
- Let a pair of dice be thrown and the random variable X be the sum of the numbers that appear on the two dice. Find the mean or expectation of X
 - Three balls are drawn without replacement from a bag containing 4 white and 5 red balls. Find the probability distribution of number of red balls.
 - There is a group of 20 persons who are rich, out of these 5 are helpful to poor people. 3 persons are selected at random, write the probability distribution of selected person who are helpful to poor. Also find the mean of the distribution.
 - There is a group of 50 people who are patriotic out of which 20 believe in non-violence. Two persons are selected at random out of them. Write the probability distribution for the selected person who are non-violent. Also find expectation of the distribution.
 - Three rotten apples are mixed with 7 fresh apples. Find the probability distribution of number of rotten apples, if three apples are drawn one by one with

replacement.

8. A biased die is twice as likely to show an even number as an odd number. The die is rolled three times. If occurrence of an even number is considered as a success, then write the probability distribution of number of successes. Also find the mean number of successes.

9. Two cards are drawn simultaneously (or successively without replacement). From a well shuffled pack of 52 cards. Find mean, variance and standard deviation of number of kings.

10. Find the mean number of heads in three tosses of a fair coin.

Solution of questions based on Probability Distribution

Ans1.

(i) The sample space of the random experiment 'a die is tossed' is $S = \{1, 2, 3, 4, 5, 6\}$. Therefore $n(S) = 6$. Let **s** denote the success i.e., getting a number greater than 4 on the dice and **f**, the failure.

$$P(s) = P(\text{a number greater than 4}) = P(\{5, 6\}) = \frac{2}{6} = \frac{1}{3}$$

$$\therefore P(f) = 1 - P(s) = 1 - \frac{1}{3} = \frac{2}{3}$$

Let **X** denote the number of successes in two tosses of a die, then X can take values 0, 1, 2.

$$P(X = 0) = P(\text{no success}) = P(\{ff\}) = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$$

$$P(X = 1) = P(\text{one success}) = P(\{sf, fs\}) = \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} = \frac{4}{9}$$

$$P(X = 2) = P(\text{two successes}) = P(\{ss\}) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

Therefore the probability distribution of X is

X	0	1	2
---	---	---	---

P(X)	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$
------	---------------	---------------	---------------

(ii) We know that when a dice is through, sample space is $S = \{1, 2, 3, 4, 5, 6\}$.
 Therefore $n(S) = 6$

Let E be the event of getting a 6 on the dice.

$$\therefore P(E) = \frac{1}{6}$$

Let **s** denote the success i.e., getting at least 6 on the dice when it is tossed two times and **f**, the failure.

$$P(s) = P(\{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,1), (6,2), (6,3), (6,4), (6,5)\})$$

$$= \frac{11}{36}$$

$$\therefore P(f) = 1 - P(s) = 1 - \frac{11}{36} = \frac{25}{36}$$

Let **X** denote the number of successes in two tosses of a die, then X can take values 0, 1.

Therefore the probability distribution of X is

X	0	1
P(X)	$\frac{25}{36}$	$\frac{11}{36}$

Ans2.

(i) We know that for a probability distribution $\sum P(X) = 1$

Or

$$P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=4) + P(X=6) + P(X=7) = 1$$

$$\therefore 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + (7k^2 + k) = 1$$

$$\text{or } 10k^2 + 9k - 1 = 0$$

$$10k^2 + 10k - 1k - 1 = 0$$

$$10k(k+1) - 1(k+1) = 0 \quad (k$$

$$+ 1)(10k - 1) = 0$$

$$\text{Either } k + 1 = 0 \text{ or } 10k - 1 = 0$$

But $k = -1$ rejected, since $P(X=1) = k$ and hence must be > 0 by def of P.D.

$$\therefore 10k - 1 = 0 \quad \text{or} \quad k = \frac{1}{10} \quad \dots\dots\dots(1)$$

$$\begin{aligned} \text{(ii)} \quad P(X < 3) &= P(X=0) + P(X=1) + P(X=2) \\ &= 0 + k + 2k = 3k = \frac{3}{10} \end{aligned}$$

$$\left[\because k = \frac{1}{10} \text{ by (1)} \right]$$

$$\begin{aligned} \text{(iii)} \quad P(X > 6) &= P(X=7) = 7k^2 + k \\ &= 7\left(\frac{1}{10}\right)^2 + \frac{1}{10} = \frac{7}{100} + \frac{1}{10} = \frac{17}{100} \quad \left[\because k = \frac{1}{10} \text{ by (1)} \right] \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad P(0 < X < 3) &= P(X=1) + P(X=2) \\ &= k + 2k = 3k = \frac{3}{10} \quad \left[\because k = \frac{1}{10} \text{ by (1)} \right] \end{aligned}$$

Ans3. The sample space of the experiment is $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), \dots, (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$.
Therefore $n(S) = 36$

Let X denote the sum of numbers obtained on two dice, then X can take values 2,3,4,5,6,7,8,9,10,11 or 12.

$$\text{Now } P(X=2) = P(\{(1,1)\}) = \frac{1}{36}$$

$$P(X=3) = P(\{(1,2), (2,1)\}) = \frac{2}{36}$$

$$P(X=4) = P(\{(1,3), (2,2), (3,1)\}) = \frac{3}{36}$$

$$P(X=5) = P(\{(1,4), (2,3), (3,2), (4,1)\}) = \frac{4}{36}$$

$$P(X=6) = P(\{(1,5), (2,4), (3,3), (4,2), (5,1)\}) = \frac{5}{36}$$

$$P(X=7) = P(\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}) = \frac{6}{36}$$

$$P(X=8) = P(\{(2,6), (3,5), (4,4), (5,3), (6,2)\}) = \frac{5}{36}$$

$$P(X=9) = P(\{(3,6),(4,5),(5,4),(6,3)\}) = \frac{4}{36}$$

$$P(X=10) = P(\{(4,6),(5,5),(6,4)\}) = \frac{3}{36}$$

$$P(X=11) = P(\{(5,6),(6,5)\}) = \frac{2}{36}$$

$$P(X=12) = P(\{(6,6)\}) = \frac{1}{36}$$

Therefore the probability distribution of X is

X	2	3	4	5	6	7	8	9	10	11	12
P(X) or p_i	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Therefore

$$\begin{aligned} \mu &= \sum x_i p_i = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + 6 \times \frac{5}{36} + 7 \times \frac{6}{36} + 8 \times \frac{5}{36} + 9 \times \frac{4}{36} \\ &\quad + 10 \times \frac{3}{36} + 11 \times \frac{2}{36} + 12 \times \frac{1}{36} \\ &= \frac{2+6+12+20+30+42+40+36+30+22+12}{36} = 7 \end{aligned}$$

Hence the mean is 7.

Ans4.

Let X denotes the number of red balls drawn, then X can take values 0,1,2,3. If W and R denote the event, drawing a white ball and red ball respectively. Then

$$P(X=0) = P(\{(WWW)\}) = \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} = \frac{24}{504} = \frac{1}{21}$$

$$P(X=1) = P(\{(RWW), (WRW), (WWR)\})$$

$$= \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} + \frac{4}{9} \times \frac{5}{8} \times \frac{3}{7} + \frac{4}{9} \times \frac{3}{8} \times \frac{5}{7} = \frac{60+60+60}{9 \times 8 \times 7} = \frac{180}{504} = \frac{5}{14}$$

$$P(X=2) = P(\{(WRR), (RWR), (RRW)\})$$

$$= \frac{4}{9} \times \frac{5}{8} \times \frac{4}{7} + \frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} + \frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} = \frac{80+80+80}{9 \times 8 \times 7} = \frac{240}{504} = \frac{10}{21}$$

$$P(X=3) = P(\{(RRR)\}) = \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} = \frac{60}{504} = \frac{5}{42}$$

Hence the probability distribution for X is

X	0	1	2	3
P(X)	$\frac{1}{21}$	$\frac{5}{14}$	$\frac{10}{21}$	$\frac{5}{42}$

Ans5. Let X denotes the number of people who helpful to poor people, then X can take values 0,1,2,3. If A and B denote the event, people who helpful to poor people and people who not helpful to poor people respectively. Then

$$P(A) = \frac{5}{20} = \frac{1}{4} \quad \text{and} \quad P(B) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(X=0) = P(\{(BBB)\}) = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64}$$

$$P(X=1) = P(\{(ABB), (BAB), (BBA)\})$$

$$= \frac{1}{4} \times \frac{3}{4} \times \frac{3}{4} + \frac{3}{4} \times \frac{1}{4} \times \frac{3}{4} + \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{27}{64}$$

$$P(X=2) = P(\{(BAA), (ABA), (AAB)\})$$

$$= \frac{3}{4} \times \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} \times \frac{3}{4} = \frac{9}{64}$$

$$P(X=3) = P(\{(AAA)\}) = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}$$

Hence the probability distribution for X is

X	0	1	2	3
P(X)	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

Therefore

$$\mu = \sum x_i p_i = 0 \times \frac{27}{64} + 1 \times \frac{27}{64} + 2 \times \frac{9}{64} + 3 \times \frac{1}{64}$$

$$= \frac{0+27+18+3}{64} = \frac{48}{64} = \frac{3}{4} = 0.75$$

Hence the mean is 0.75.

Ans6. Let X denotes the number of people who patriotic and believe in non-violence, then X can take values 0, 1, 2. If A and B denote the people who believe in non-violence and people who believe in violence respectively.

$$\text{Then } P(A) = \frac{20}{50} = \frac{2}{5} \quad \text{and } P(B) = 1 - \frac{2}{5} = \frac{3}{5}$$

$$P(X=0) = P(\{(BB)\}) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$$

$$P(X=1) = P(\{(AB), (BA)\}) \\ = \frac{2}{5} \times \frac{3}{5} + \frac{3}{5} \times \frac{2}{5} = \frac{12}{25}$$

$$P(X=2) = P(\{(AA)\}) \\ = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$$

Hence the probability distribution for X is

X	0	1	2
P(X)	$\frac{9}{25}$	$\frac{12}{25}$	$\frac{4}{25}$

Therefore

$$\mu = \sum x_i p_i = 0 \times \frac{9}{25} + 1 \times \frac{12}{25} + 2 \times \frac{4}{25} \\ = \frac{0+12+8}{25} = \frac{20}{25} = \frac{4}{5} = 0.8$$

Hence the mean is 0.8.

Ans7. Let X denotes the number of rotten apples, then X can take values 0,1,2,3. If A and B denote the drawn apple is rotten and drawn apple is fresh respectively.

$$\text{Then } P(A) = \frac{3}{10}$$

$$\text{and } P(B) = 1 - \frac{3}{10} = \frac{7}{10}$$

$$\frac{7}{10} \times \frac{7}{10} \times \frac{7}{10} = \frac{343}{1000}$$

(BAB), (BBA))

$$P(X=0) = P(\{(BBB)\}) =$$

$$P(X=1) = P(\{(ABB),$$

$$= \frac{3}{10} \times \frac{7}{10} \times \frac{7}{10} + \frac{7}{10} \times \frac{3}{10} \times \frac{7}{10} + \frac{7}{10} \times \frac{7}{10} \times \frac{3}{10} = \frac{441}{1000}$$

$$P(X=2) = P(\{(BAA), (ABA), (AAB)\})$$

$$= \frac{7}{10} \times \frac{3}{10} \times \frac{3}{10} + \frac{3}{10} \times \frac{7}{10} \times \frac{3}{10} + \frac{3}{10} \times \frac{3}{10} \times \frac{7}{10} = \frac{189}{1000}$$

$$P(X=3) = P(\{(AAA)\}) = \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10} = \frac{27}{1000}$$

Hence the probability distribution for X is

X	0	1	2	3
P(X)	$\frac{343}{1000}$	$\frac{441}{1000}$	$\frac{189}{1000}$	$\frac{27}{1000}$

Ans8. Let A denote the event 'occurrence of an even number' and B denote the 'occurrence of an odd number' if a dice is through.

According to question $P(A) = 2 P(B)$. But there are 3 even and 3 odd numbers so

$$P(A) + P(B) = 1 \text{ or } 2P(B) + P(B) = 1 \text{ or } 3 P(B) = 1 \text{ or } P(B) = \frac{1}{3} \text{ and } P(A) =$$

$$\frac{2}{3}$$

Let X denotes the number of occurrence of an even number when a dice is rolled, then X can take values 0,1,2,3.

$$P(X=0) = P(\{(BBB)\}) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

$$P(X=1) = P(\{(ABB), (BAB), (BBA)\})$$

$$= \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{6}{27}$$

$$P(X=2) = P(\{(BAA), (ABA), (AAB)\})$$

$$= \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{12}{27}$$

$$P(X=3) = P(\{(AAA)\}) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$$

Hence the probability distribution for X is

X	0	1	2	3
P(X)	$\frac{1}{27}$	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{8}{27}$

Therefore $\mu = \sum x_i p_i = 0 \times \frac{1}{27} + 1 \times \frac{6}{27} + 2 \times \frac{12}{27} + 3 \times \frac{8}{27}$

$$= \frac{0+6+24+24}{27} = \frac{54}{27} = 2$$

Hence the mean is 2.

Ans9. Let X denotes the number of king drawn from a well shuffled deck of 52 cards, then X can take values 0,1,2.

If A and B denote the event that the card drawn is a king and B the event that the card drawn is not a king respectively.

$$P(X=0) = P(\{(BB)\}) = \frac{48}{52} \times \frac{47}{51}$$

$$= \frac{2256}{2652} = \frac{188}{221} \quad \{ \text{since the first card is not replaced} \}$$

$$P(X=1) = P(\{(AB), (BA)\})$$

$$= \frac{4}{52} \times \frac{48}{51} + \frac{48}{52} \times \frac{4}{51} = \frac{32}{221}$$

$$P(X=2) = P(\{(AA)\})$$

$$= \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

Hence the probability distribution for X is

X	0	1	2
P(X)	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

Therefore

$$\text{mean } (\mu) = \sum x_i p_i = 0 \times \frac{188}{221} + 1 \times \frac{32}{221} + 2 \times \frac{1}{221} = \frac{34}{221} = \frac{2}{13}$$

$$\text{Variance} = \sum (x_i^2) p_i - (\mu)^2 = \left\{ 0^2 \times \frac{188}{221} + 1^2 \times \frac{32}{221} + 2^2 \times \frac{1}{221} \right\} - \left(\frac{2}{13} \right)^2 = \frac{36}{221} - \frac{4}{169} = \frac{400}{2873}$$

$$\text{Standard deviation} = \sqrt{\text{Variance}} = \sqrt{\frac{400}{2873}}$$

Ans10. Let X denotes the number of heads in the three tosses of a fair coin, then X can take values 0,1,2,3. If H and T denote the event getting a head when we toss a coin and the event getting a Tell respectively. $P(H) = \frac{1}{2}$

and $P(T) = \frac{1}{2}$

$$P(X=0) = P(\{(TTT)\}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$$P(X=1) = P(\{(HTT), (THT), (TTH)\})$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{8}$$

$$P(X=2) = P(\{(THH), (HTH), (HHT)\})$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{8}$$

$$P(X=3) = P(\{(HHH)\}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Hence the probability distribution for X is

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Therefore

$$\text{mean } (\mu) = \sum x_i p_i = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{12}{8} = \frac{3}{2}$$

BINOMIAL DISTRIBUTION

1. If a fair coin is tossed 10 times, find the probability of

- a) Exactly six heads
 - b) At least 6 heads
 - c) At most six heads
2. A pair of dice is thrown four times. If getting a doublet is considered as success, find probability of two success.
 3. A box containing 100 bulbs, 10 are defective find the probability that out of assemble of 5 bulbs none is defective.
 4. The probability that a student is not a swimmer is $1/5$. Find the probability that out of 6 students 4 are swimmers.
 5. Find the mean of the binomial distribution $B(4, 1/3)$
 6. The probability of a shooter hitting a target is $3/4$. How many minimum number of times must he fire so that the probability of hitting the target at least once is more than 0.99?
 7. How many times must a man toss a fair coin so that the probability of having at least one head is more than 90%?
 8. It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such article 9 are defective?
 9. A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is $1/100$. What is the probability that he will win a prize (a) at least once (b) exactly once (c) at least twice.
 10. Suppose X has a binomial distribution $B(6, 1/2)$. Show that $X=3$ is the most likely outcomes.
-

MULTIPLICATION THEOREM ON PROBABILITY/INDEPENDENT EVENTS

1. A fair die tossed thrice. Find the probability of getting an odd number at least once.
2. Probability of solving specific problem independently by A and B are $1/2$ and $1/3$ respectively. If both try to solve the problem independently, find the probability that (1) the problem is solved (2) exactly one of them solved the problem
3. Event A and B are such that $P(A) = 1/2$, $P(B) = 7/12$ and $P(\text{not } A \text{ or not } B) = 1/4$. State whether A and B are independent.
4. Two cards are drawn at random and without replacement from a pack of 52 cards.

Find the probability that both the cards are black.

5. The probabilities of A, B and C hitting a target are $\frac{1}{3}$, $\frac{2}{7}$ and $\frac{3}{8}$ respectively. If all the three tried to shoot the target simultaneously, find the probability that exactly one of them can shoot it.
6. A husband and a wife appear for interview for two vacancies for the same post. The probability of husband's selection is $\frac{1}{7}$ and that of wife's is $\frac{1}{5}$. What is the probability that (1) only one of them will be selected (2) at least one of them will be selected.
7. In a hurdle race a player has to cross 10 hurdles. The probability that he will clear a hurdle is $\frac{5}{6}$. What is the probability that he will clear fewer than 2 hurdles?
8. A coin is tossed once. If it shows a head, it is tossed again but if it shows a tail, then a die is thrown. If all possible outcomes are equally likely, find the probability that the die shows a number greater than 4, if it is known that the first throw of coin results a tail.
9. Three persons A B and C fire a target in turn, starting with A. Their probability of hitting the target are 0.5, 0.3 and 0.2 respectively. Find the probability of at most one hit.
10. Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that (1) both balls are red (2) one of them is black and other is red.

MODEL ANSWERS:

BIONOMIAL DISTRIBUTION

1. $n=10, p=\frac{1}{2}$

$$P(X=x) = {}^nC_x q^{n-x} p^x$$

$$(i) P(X=6) = {}^{10}C_6 \left(\frac{1}{2}\right)^{10} = 105/512$$

$$(ii) P(X \geq 6) = P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= {}^{10}C_6 \left(\frac{1}{2}\right)^{10} + {}^{10}C_7 \left(\frac{1}{2}\right)^{10} + {}^{10}C_8 \left(\frac{1}{2}\right)^{10} + {}^{10}C_9 \left(\frac{1}{2}\right)^{10} + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10}$$

$$= 193/512$$

$$(iii) P(X \leq 6) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$$

$$=10C_0(1/2)^{10}+10C_1(1/2)^{10}+10C_2(1/2)^{10}+10C_3(1/2)^{10}+10C_4(1/2)^{10}+10C_5(1/2)^{10}+10C_6(1/2)^{10}$$

$$=53/64$$

2. $n=4, p = \frac{6}{36} = \frac{1}{6} \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$$q = 1 - \frac{1}{6} = \frac{5}{6}$$

$$P(X=x) = nC_x q^{n-x} p^x$$

$$P(X=2) = 4C_2 (5/6)^2 (1/6)^2$$

$$=25/216$$

3. $n=5, p = 10/100 = 1/10$

$$q = 1 - 1/10 = 9/10$$

$$P(X=0) = q^n = (9/10)^5$$

4. $n=5,$

probability that a student is not a swimmer $= 1/5$

$$p = P(\text{a student can swim}) = 1 - 1/5 = 4/5$$

$$q = 1/5$$

$$P(X=4) = 5C_4 (1/5) (4/5)^4$$

5. $n=4, p=1/3, q = 1 - 1/3 = 2/3$

$$P(X=x) = nC_x q^{n-x} p^x$$

x_i	$P(x_i)$	$x_i P(x_i)$
0	$4C_0 (2/3)^4$	0
1	$4C_1 (2/3)^3 (1/3)$	32/81
2	$4C_2 (2/3)^2 (1/3)^2$	48/81

3	$4C_3(2/3) (1/3)^3$	24/81
4	$4C_4(1/3)^4$	4/81
Total		108/81

$$\text{Mean} = 108/81 = 4/3$$

6. $p = 3/4$

$$q = 1/4$$

$$P(X=x) = nC_x q^{n-x} p^x$$

$$P(X=x) = nC_x (1/4)^{n-x} (3/4)^x$$

$$P(\text{Hitting the target at least once}) > 0.99$$

$$\text{ie, } P(x \geq 1) > 0.99$$

$$\text{ie, } 1 - P(x=0) > 0.99$$

$$\text{ie, } 1 - nC_0 (1/4)^n > 0.99$$

$$\text{ie,, } (1/4)^n < 0.01$$

$$\text{ie, } 4^n > 100$$

The minimum value of n to satisfy the inequality is 4

Therefore the shooter must fire 4 times.

7. $p = 1/2$

$$q = 1/2$$

$$P(X=x) = nC_x q^{n-x} p^x$$

$$P(X=x) = nC_x (1/2)^{n-x} (1/2)^x$$

$$P(X=x) = nC_x (1/2)^n$$

$$P(\text{at least one head}) > 0.9$$

$$\text{ie, } P(x \geq 1) > 0.9$$

$$\text{ie, } 1 - P(x=0) > 0.9$$

$$\text{ie, } 1 - nC_0 (1/2)^n > 0.9$$

$$\text{ie,, } (1/2)^n < 0.1$$

$$\text{ie, } 2^n > 10$$

The minimum value of n to satisfy the inequality is 4

The man must toss the coin at least 4 times.

$$8. \ n=12, \ p=10/100 = 1/10$$

$$q = 1 - 1/10 = 9/10$$

$$P(X=x) = nC_x q^{n-x} p^x$$

$$P(X=9) = 12C_9 (9/10)^3 (1/10)^x$$

$$= 22 \times 9^3 / 10^{11}$$

$$9. \ p = 1/100 \ q = 99/100, \ n = 50$$

$$P(X=x) = nC_x q^{n-x} p^x$$

$$P(\text{winning atleast once}) = 1 - P(\text{not winning})$$

$$= 1 - P(x=0)$$

$$= 1 - 50C_0 (99/100)^{50}$$

$$= 1 - (99/100)^{50}$$

$$P(\text{exactly once}) = P(X=1)$$

$$= {}^{50}C_1 (99/100)^{49} (1/100)$$

$$= \frac{1}{2} (99/100)^{49}$$

$$P(\text{atleast twice}) = 1 - P(X=0) - P(X=1)$$

$$1 - (1 - (99/100)^{50} + \frac{1}{2} (99/100)^{49})$$

$$1 - 149/100 (99/100)^{49}$$

10. $n=6, p=1/2, q=1/2$

$$P(X=x) = {}^nC_x q^{n-x} p^x$$

x_i	$P(x_i)$
0	${}^6C_0 (1/2)^6 = 1/64$
1	${}^6C_1 (1/2)^6 = 6/64$
2	${}^6C_2 (1/2)^6 = 15/64$
3	${}^6C_3 (1/2)^6 = 20/64$
4	${}^6C_4 (1/2)^6 = 15/64$
5	${}^6C_5 (1/2)^6 = 6/64$
6	${}^6C_6 (1/2)^6 = 1/64$

Clearly $P(X=3)$ is maximum. Therefore $X=3$ is the most likely outcome.

